Unit - I

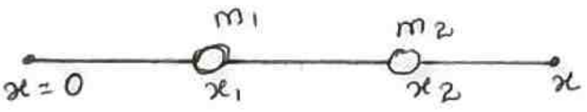
Mechanics

Multiparticle dynamics : Center of mass (CM) - CM of Continuous bodies - motion of the CM - Kinetic energy of system of Particles. Rotation of rigid bodies ; Rotational kinematics - rotational kinetic energy and moment of Inertia - theorems of M.I - moment of Inertia of continuous bodies - M.I of a diatomic molecule - torque - rotational dynamics of rigid bodies - Conservation of angular momentum - rotational energy state of a rigid diatomic molecule - gyroscope - Torsional pendulum - double pendulum - Introduction to non-linear oscillations.

① Derive an expression for Multiparticle Dynamics

Definition

A mechanical system consists of two or more particles is called multiparticle system.



Consider two particles of mass m_1 and m_2 moving in one-dimension with co-ordinates x_1 and x_2.

$\vec{F_1}$ - force acting on body m_1

According to Newton's law

$$m_1 \vec{a_1} = \vec{F_1}$$

$$m_1 \frac{d^2 x_1}{dt^2} = F_1 \quad (or)$$

$$m_1 \ddot{\vec{x_1}} = F_1 \longrightarrow (1)$$

$\frac{d^2 x_1}{dt^2}$ is denoted as $\ddot{\vec{x_1}}$ $\Big|$ $\vec{a} = \frac{d^2 \vec{x_1}}{dt^2}$

Force $\vec{F_1}$ can be divided into two Parts

$$\vec{F_1} = \vec{F_{12}} + \vec{F_{1e}} \longrightarrow (2)$$

where

$$\vec{F_1} = \vec{F_{12}} + \vec{F_{1e}}$$

F_{1e} - Sum of all external force

eqn (1) becomes

$$m_1 \ddot{\vec{x_1}} = \vec{F_{12}} + \vec{F_{1e}}$$

For particle 2,

$$m_2 \vec{a_2} = \vec{F_2}$$

$$m_2 \frac{d^2 x_2}{dt^2} = m_2 \ddot{\vec{x_2}}$$

$$\vec{F_2} = F_{21} + \vec{F_{2e}}$$

Total force on system of two particle is given by eqns (3) and (4)

$$m_1\vec{\ddot{x}}_1 + m_2\vec{\ddot{x}}_2 = \vec{F}_{12} + \vec{F}_{21} + \vec{F}_{1e} + \vec{F}_{2e}$$

Newton's third law $\vec{F}_{12} = -\vec{F}_{21}$

$$m_1\vec{\ddot{x}}_1 + m_2\vec{\ddot{x}}_2 = \vec{F}_{12} + (-\vec{F}_{12}) + \vec{F}_{1e} + \vec{F}_{2e}$$

$$m_1\vec{\ddot{x}}_1 + m_2\vec{\ddot{x}}_2 = \vec{F}_{1e} + \vec{F}_{2e} = \vec{F}_e$$

where, $\vec{F}_e$ - net external force

Total mass of the system is equal to

$$M = m_1 + m_2$$

Multiplying and dividing eqn (7) by M in L.H.S

$$M\left(\frac{m_1\vec{\ddot{x}}_1 + m_2\vec{\ddot{x}}_2}{M}\right) = \vec{F}_e$$

(or)

$$M\vec{\ddot{x}} = \vec{F}_e$$

$$M\left(\frac{m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2}}{M}\right) = \vec{F}_e$$

$$M\frac{d^2}{dt^2}\left(\frac{m_1x_1 + m_2x_2}{M}\right) = \vec{F}_e$$

$$M\frac{d^2x}{dt^2} = \vec{F}_e \quad \text{(or)} \quad M\vec{\ddot{x}} = \vec{F}_e$$

where

$$\boxed{x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}} \longrightarrow (11)$$

eqn (11) called the Center of mass (CM)

CM has a location x which is a weighted average of x_1 and x_2

$$x = \frac{m_1x_1}{m_1+m_2} + \frac{m_2x_2}{m_1+m_2} \longrightarrow (12)$$

If $m_1 = m_2 = m$, then $M = 2m$

$$x = \frac{x_1 + x_2}{2} \longrightarrow (13)$$

In general, the CM for a system of N number of particles is obtained by extending eqn (1)

$$x = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_Nx_N}{m_1 + m_2 + m_3 + \cdots m_N}$$

$$x = \frac{\sum_{i=1}^{N} m_ix_i}{\sum m_i} \quad \text{(or)}$$

$$x = \frac{\sum_{i=1}^{N} m_ix_i}{M} \quad \left(\sum m_i = M\right)$$

For continuous distribution of masses, for example in one dimension, the CM is represented as

$$x = \lim_{\Delta m \to 0} \frac{\sum_{i=1}^{N} \Delta m_i x_i}{M}$$

Summation is changed into integration

$$x = \frac{\int x\,dm}{\int dm} \quad \left(\int dm = M\right)$$

$$\boxed{x = \frac{\int x\,dm}{M}}$$

<u>Example</u> :

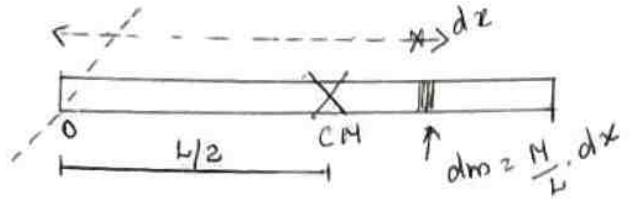
In the case of a uniform rod

Mass per unit length of the rod $= \frac{M}{L}$

Mass of the elemental length dx of the rod $dx = \left(\frac{M}{L}\right)$

between limits $x=0$ and $x=L$

$$x = \frac{\int_0^L x\frac{M}{L}dx}{M} = \left|\frac{1}{L}\left(\frac{x^2}{2}\right)\right|_0^L = \frac{L}{2}$$



For N particles in three dimensions (3D)

For x direction, $x_{CM} = \frac{\sum_i m_i x_i}{M}$

For y-direction, $y_{CM} = \frac{\sum_i m_i y_i}{M}$

For z-direction, $z_{CM} = \frac{\sum_i m_i z_i}{M}$

Thus in vector notation

$\vec{r_i} = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$

So the position of the CM is

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r_i}}{M}$$

and for continuous object

$$\boxed{\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\, dm}$$

(2) Write short notes on Centre of Mass (CM).

When the body rotates or vibrates during translatory motion then its motion can be represented by a point that moves in the same way as that of a single particle subjected to the same external forces would move. This point is called centre of mass of a system

Definition:

A point in the system at which whole mass of the body is supposed to be concentrated is called centre of mass of the body

Examples for motion of Center of mass

i) Motion of planets and its Satellite

- Consider motion of the centre of mass of the earth and moon system.
- Moon moves round the earth in a circular orbit.
- Earth moves round the sun in an elliptical orbit.
- Both earth and moon move in circular orbits about their common centre.
- Mutual gravitational attractions are the internal forces.
- Earth and moon are the external forces acting on the centre of the system.

ii) Projectile Trajectory

- When a cracker is fired at an angle with the horizontal it explodes in the air.
- Different pieces of the cracker follows different Parabolic Paths.

iii) Decay of a Nucleus

- Consider a spontaneous decay of radioactive nucleus into two fragments.
- obey the laws of conservation of energy and momentum
- Nucleus decays under the effect of internal forces.

Centre of mass of two point masses.

Following three ways based on the choice of the coordinate system

i) when the masses are on positive x-axis

- The origin is taken arbitrarily.
- m_1, m_2 - masses, x_1, x_2 - positions

$$\boxed{X_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

ii) when the origin coincides with any one of the masses

$$X_{CM} = \frac{m_1(0) + m_2 x_2}{m_1 + m_2}$$

The equation further simplified as

$$\boxed{X_{CM} = \frac{m_2 x_2}{m_1 + m_2}}$$

iii) when the origin coincides with the centre of mass itself

$$0 = \frac{m_1(-x_1) + m_2 x_2}{m_1 + m_2}$$

$$0 = m_1(-x_1) + m_2 x_2$$

$$\boxed{m_1 x_1 = m_2 x_2}$$

The above eqn is known as Principle of moments.

③ Write notes in motion of centre of mass

When a rigid body moves, it centre of mass will also move along the body.

Like velocity $(\vec{v}_{CM})$ and acceleration $(\vec{a}_{CM})$ for the centre of mass, we can differentiate the expression for position of centre of mass with respect to time once and twice respectively.

Let us take the motion along x direction only

$$\vec{v}_{CM} = \frac{d\vec{x}_{cm}}{dt} \frac{d}{dt}\left(\frac{\Sigma m_i x_i}{\Sigma m_i}\right)$$

$$= \frac{\sum m_i \left(\frac{d\vec{x_i}}{dt}\right)}{\sum m_i}$$

$$\vec{V}_{CM} = \frac{\sum m_i \vec{v_i}}{\sum m_i} \longrightarrow (1)$$

$$\vec{a}_{cm} = \frac{d}{dt}\left(\frac{d\vec{x}_{cm}}{dt}\right)$$

$$= \left(\frac{d\vec{v}_{cm}}{dt}\right)$$

$$= \frac{\sum m_i \left(\frac{d\vec{v_i}}{dt}\right)}{\sum m_i}$$

$$\boxed{\vec{a}_{cm} = \frac{\sum m_i \vec{a_i}}{\sum m_i}} \longrightarrow (2)$$

In the absence of external force $\vec{F}_{ext} = 0$, the individual rigid bodies of a system can move or shift only due to the internal forces.

From eqns (1) and (2)

$$\vec{V}_{cm} = \frac{\sum m_i \vec{V_i}}{\sum m_i} = 0 \quad (or)$$

$$\vec{V}_{cm} = \text{constant}$$

It shows that

$$\vec{a}_{cm} = \frac{\sum m_i \vec{a_i}}{\sum m_i} = 0$$

The individual particles may still move with the respective velocities and accelerations due to internal forces.

In the presence of external force $(\vec{F}_{ext} \neq 0)$, the centre of mass of the system will accelerate as given the following eqn

$$\vec{F}_{ext} = (\sum m_i)\, \vec{a}_{cm}$$

$$\vec{F}_{ext} = M \vec{a}_{cm} \quad (\sum m_i = M)$$

$$\boxed{\vec{a}_{cm} = \frac{\vec{F}_{ext}}{M}}$$

4) Describe about centre of mass (CM) of continuous Bodies (Rigid bodies)

Experimental location of the centre of mass

- Centre of masses of homogeneous
- Regular shaped bodies coincides
- If the body is of irregular shape the location of its centre of mass is difficult.
- The centre of mass is found to coincide with centre of gravity of the bodies.
- It can be easily obtained either by pivoting the body to a balanced position or suspending it from some fixed point in it.
- Both are similar methods
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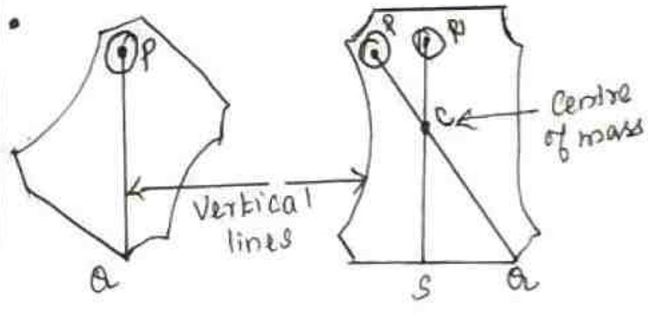
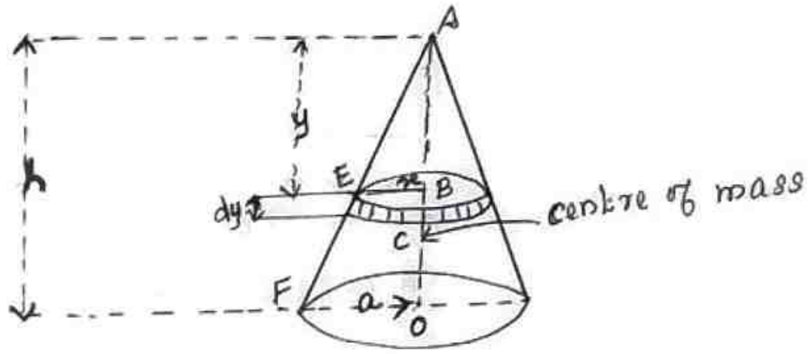


• The body first hung from some (any) points P and a vertical line PQ is drawn when the body is in equilibrium.
• The body is then hung from some other point R and a vertical line RS is drawn.
• The point of intersection C of these two lines PQ and RS gives the position of centre of mass.

Expression for a centre of mass of a solid cone



a - radius
h - height
ρ - density

If the solid cone is homogeneous its mass

$$m = \frac{1}{3}\pi a^2 h . \rho$$

The centre of mass lies on the axis of symmetry AO.

dy - thickness

Elementary disc of radius x, y - distance

The mass of elementary disc is

$$dm = \rho(\pi x^2)\,dy \rightarrow (1)$$

$\frac{x}{a} = \frac{y}{h}$ [In similar triangles AEB and AFO]

$$x = \frac{a}{h} y$$

$$\therefore dm = \rho\pi\left(\frac{a}{h} . y\right)^2 . dy \rightarrow (2)$$

from eqn (1)

$$Y_{CM} = \frac{1}{M}\int y\,dm \rightarrow (3)$$

on substituting the value of dm, we get

$$Y_{cm} = \frac{1}{M}\int_0^h y\rho\pi\left(\frac{a}{h}y\right)^2 dy$$

(or)

$$Y_{cm} = \frac{\rho\pi a^2}{Mh^2}\int_0^h y^3 dy \rightarrow (4)$$

limits $y = 0$ to $y = h$

$$Y_{cm} = \frac{\rho\pi a^2}{Mh^2}\left[\frac{y^4}{4}\right]_0^h$$

$$= \frac{\rho\pi a^2}{M\cancel{h^2}}\frac{\cancel{h^4}}{4} = \frac{\rho\pi a^2 h^2}{4M} \rightarrow (5)$$

M = Total mass of the solid cone $= \frac{1}{3}\pi a^2 h\rho$

$$Y_{CM} = \frac{\rho\pi a^2 h^2 3}{4\pi a^2 h\rho}$$

The centre of mass of cone from its vertex Y_{CM} is written as R_{CM}

$$\boxed{R_{CM} = \frac{3}{4}h}$$

(5) Derive an expression for kinetic energy of the system of particles

- Let n- number of particles in a system.
- i^{th} particle of this system depends on the external force $\vec{F_i}$.
- kinetic energy be

$$E_{Ki} = 1/2\, m V_i^2$$

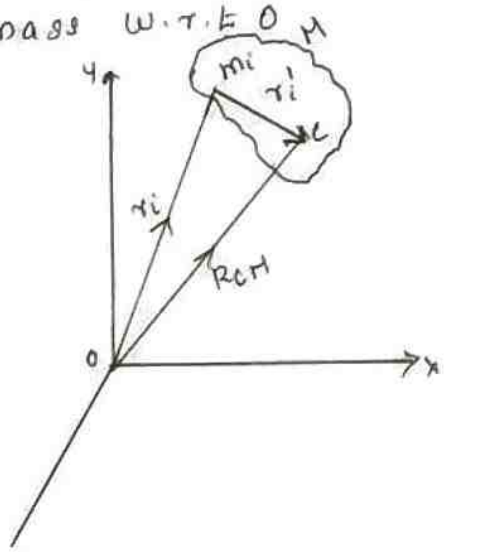
$$E_{Ki} = 1/2\, m(v_i \cdot v_i) \longrightarrow (i)$$

$\vec{r_i}$ - Position vector of the i^{th} particle w.r.t O.

$\vec{r_i'}$ - Position vector of the centre of mass with respect to $\vec{r_i}$

$$\boxed{\vec{r_i} = \vec{r_i'} + \vec{R_{CM}}} \longrightarrow (2)$$

$\vec{R_{CM}}$ - Position vector of centre of mass w.r.t O



Differentiating eqn (2)

$$\frac{d\vec{r_i}}{dt} = \frac{d\vec{r_i'}}{dt} + \frac{d\vec{R_{CM}}}{dt} \quad (\text{or})$$

$$v_i = v_i' + v_{cm} \longrightarrow (3)$$

Putting eqn (3) in (1)

$$E_{Ki} = 1/2\, m_i \left[(v_i' + v_{cm}) \cdot (v_i' + v_{cm}) \right]$$

$$= 1/2\, m_i \left[(v_i'^2 + 2 v_i' \cdot v_{cm} + v_{cm}^2) \right]$$

$$E_{Ki} = \frac{1}{2} m_i v_i'^2 + \frac{2}{2} m_i v_i' \cdot v_{cm} + 1/2\, m_i v_{cm}^2$$

$$= 1/2\, m_i v_i'^2 + m_i v_i' \cdot v_{cm} + 1/2\, m_i v_{cm}^2 \longrightarrow (4)$$

The sum of K.E of all the particles can be obtained from eqn (4)

$$E_K = \sum_{i=1}^{n} E_{Ki} = \sum_{i=1}^{n} \left[\frac{1}{2} m_i v_i'^2 + m_i v_i' \cdot v_{cm} + 1/2\, m_i v_{cm}^2 \right]$$

$$E_K = \sum_{i=1}^{n} 1/2\, m_i v_i'^2 + \sum_{i=1}^{n} m_i v_i' \cdot v_{cm} + \sum_{i=1}^{n} 1/2\, m_i v_{cm}^2$$

$$E_K = 1/2\, v_{cm}^2 \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} m_i v_i'^2 + v_{cm} \sum_{i=1}^{n} m_i v_i'$$

$$E_K = 1/2\, v_{cm}^2 M + \sum_{i=1}^{n} 1/2\, m_i v_i'^2 + v_{cm} \frac{d}{dt} \sum_{i=1}^{n} m_i \vec{r_i'} \longrightarrow (5)$$

Now last term in eqn (5) is equal to zero

$$\sum_{i=1}^{n} m_i \vec{r_i'} = 0 \qquad \left(v_i' = \frac{dr_i'}{dt} \right)$$

$$\sum_{i=1}^{n} m_i \vec{r_i'} = \sum_{i=1}^{n} m_i (\vec{r_i} - R_{CM})$$

$$= \sum_{i=1}^{n} m_i \vec{r_i} - \sum_{i=1}^{m} m_i \vec{R_{CM}} \qquad (\because r_i' = r - R_{CM})$$

$$= M R_{CM} - M R_{CM} = 0 \qquad \left(\because \sum m_i \vec{r_i} = M R_{CM}\right)$$

K.E of the system of particles

$$E_K = 1/2\, M v_{cm}^2 + 1/2 \sum_{i=1}^{n} m_i v_i'^2 = E_{KCM} + E_K' \longrightarrow (6)$$

where $\boxed{E_{kcm} = 1/2\, v_{CM}^2\, M}$

$$E_k' = \sum_{i=1}^{n} 1/2\, m_i\, \vec{v_i'}^2 \longrightarrow (7)$$

It is the K.E of the system of particle w.r.t the centre of mass.

(6) Define rigid body. Derive and Explain Rotational Motion

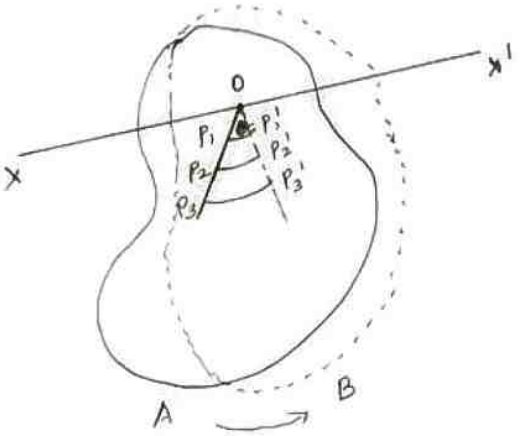
Rigid body

A rigid body is defined as that body which does not undergo any change in shape or volume when external force are applied on it.

Rotational Motion

When a body rotates about a fixed axis its motion is known as rotatory motion.

- r - radius vector, θ - angular displacement



Consider a rigid body that rotates about a fixed axis xox' passing through O and perpendicular to the Plane of paper.

- Body rotate from the position A to the position B.
- Different particles at $P_1, P_2, P_3 \ldots$ and body covers unequal distances $P_1P_1', P_2P_2', P_3P_3' \ldots$ in the same interval of time.
- Linear velocities are different.
- Angular velocity same.

- In rotational motion, different constituent of particles have different linear velocities but all of them have the same angular velocity.

Eqn of rotational Motion

- Particle start rotating with angular velocity ω_0
- angular acceleration α
- At any instant t, ω be the angular velocity of the particle

$\theta \rightarrow$ angular displacement

change in angular velocity in time

$$t = \omega - \omega_0$$

Angular acceleration = $\frac{\text{Change in angular velocity}}{\text{time taken}}$

$$\alpha = \frac{\omega - \omega_0}{t} \rightarrow (1)$$

$$\alpha t = \omega - \omega_0 \quad (or)$$

$$\omega = \omega_0 + \alpha t \rightarrow (2)$$

Average angular velocity $= \left(\frac{\omega + \omega_0}{2}\right)$

Total angular displacement

= average angular velocity × time taken

$$\theta = \left(\frac{\omega + \omega_0}{2}\right) t \rightarrow (3)$$

Substituting ω from the eqn(2)

$$\theta = \left(\frac{\omega_0 + \alpha t + \omega_0}{2}\right) t$$

$$a = \left(\frac{2\omega_0 + \alpha t}{2}\right) = \left(\frac{2\omega_0}{2} + \frac{\alpha t}{2}\right) t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \rightarrow (4)$$

From eqn(1), $t = \left(\frac{\omega - \omega_0}{\alpha}\right) \rightarrow (5)$

using eqn(5) in (3)

$$\theta = \left(\frac{\omega + \omega_0}{2}\right)\left(\frac{\omega - \omega_0}{\alpha}\right)$$

$$\theta = \left(\frac{\omega^2 - \omega_0^2}{2\alpha}\right)$$

$$2\alpha\theta = \omega^2 - \omega_0^2$$

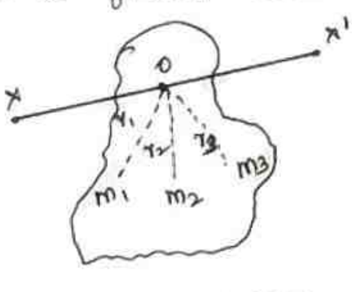
$$\omega^2 = \omega_0^2 + 2\alpha\theta \rightarrow (6)$$

The eqns (2), (4), (6) are the eqns of rotational motion

(4) Derive and Discuss about Rotational kinetic Energy and Moment of inertia

i) Rotational K.E

- Consider a rigid body, a large number of particles rotating about a fixed axis xox'



m_1, m_2, m_3 - masses

r_1, r_2, r_3 - distances

K.E of the first particle $= \frac{1}{2} m_1 v_1^2$

$$= \frac{1}{2} m_1 (r_1 \omega^2)$$

$v = r\omega$

$v_1 = r_1\omega$

K.E of the Second particle

$$= \frac{1}{2} m_2 r_2^2 \omega^2$$

K.E of the third particle

$$= \frac{1}{2} m_3 r_3^2 \omega^2$$

- Sum of the K.E

$$E_K = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2$$

$$= \frac{1}{2}\omega^2 (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2)$$

$$E_K = \frac{1}{2}\omega^2 \sum m r^2$$

$$\boxed{I = \sum m r^2}$$

$\sum m r^2$ - Moment of Inertia of a body.

$$E_K = \frac{1}{2}\omega^2 I$$

$$\boxed{E_K = \frac{1}{2} I \omega^2}$$

ii) Moment of Inertia or Rotational Motion

The Property of a body by virtue of which its opposes any change in its state of rotation about an axis is called the moment of Inertia

Moment of Inertia of a Particle

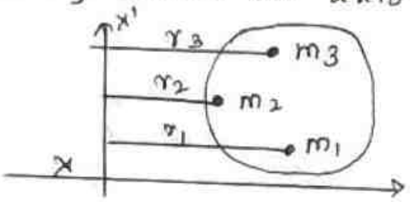
m - mass of the particle

r - distance of the particle from the axis of rotation

The moment of Inertia of the Particle

$$\boxed{I = mr^2}$$

Consider a rigid body of mass M, rotating about an axis xx'



$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots$$

$$I = \Sigma m r^2$$

Angular velocity $\omega = 1$ radian/sec

Rotational K.E $= E_R = \frac{1}{2} I \omega^2$

$$= \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} I . 1^2$$

$$E_R = \frac{1}{2} I$$

$$2E_R = I$$

$$\boxed{I = 2E_R}$$

Statement of M.I

The M.I of a particle about an axis is defined as the product of the mass of the particle and square of the distance of the particle from the axis of rotation

Significance:

- Measure of Inertia for a given system in rotational motion.
- Moment of Inertia will be greatest when the mass is farthest from the axis of rotation.

(8) Discuss the Theorems on Moment of Inertia

There are two important theorems. They are

i) Parallel axis theorem

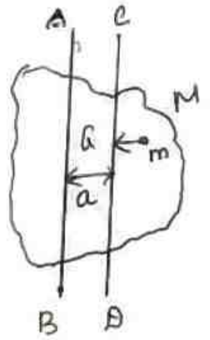
ii) Perpendicular axis theorem

i) Parallel axis theorem

Statement

The moment of inertia of a rigid body about any axis is equal to the sum of its moment of inertia about a parallel axis through its centre of gravity and the product of the mass of the body and the square of the distance between the two parallel axes.

Explanation



- G - Centre of gravity of a rigid body
- M - mass
- AB parallel to CD

I and I_G - Moment of Inertia

$$\boxed{I = I_G + Ma^2}$$

M.I of the whole body about CD, $I_G = \sum m r^2$

M.I of the particle about the axis AB $= m(r+a)^2$

M.I of the whole body about AB, $I = \sum m(r+a)^2$

$$I = \sum m(r^2 + a^2 + 2ar)$$

$$I = I_G + Ma^2 + 2a\sum mr$$

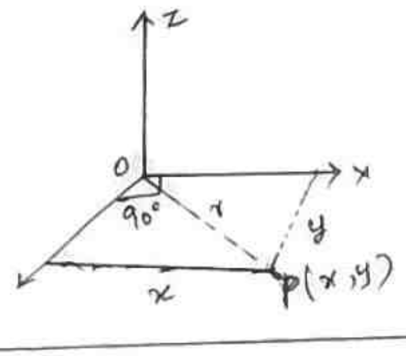
The body always balances about an axis through its centre of gravity. $\sum mr$ be zero.

$$\boxed{I = I_G + Ma^2}$$

ii) Perpendicular axis theorem

Statement

It states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the plane lamina about any two mutually perpendicular axes in its own plane and intersecting each other at the point where the perpendicular axis passes through it.



OX, OY - two mutually perpendicular axis in the plane of the lamina, intersecting each other at the point O.

OZ - Perpendicular to both OX and OY.

I_x, I_y - Moments of Inertia of the lamina about the axis OX and OY.

I_z - Moment of Inertia about the axis OZ.

$$\boxed{I_z = I_x + I_y}$$

Proof :

Consider the axes OX and OY

Moment of Inertia about OX $= my^2$

Moment of the entire lamina OX,

$$I_x = \sum my^2$$

Moment of the lamina about OY,

$$I_y = \sum mx^2$$

$$I_z = \sum mr^2 \longrightarrow (1)$$

$$r^2 = x^2 + y^2 \longrightarrow (2)$$

Substituting eqn (2) in eqn (1)

$$I_z = \sum m(x^2 + y^2)$$

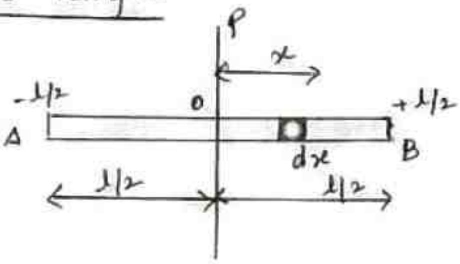
$$I_z = \sum mx^2 + \sum my^2$$

$$I_z = I_y + I_x$$

$$\boxed{I_z = I_x + I_y}$$

9) Derive an expression for moment of inertia of a thin uniform rod.

1) a) About an axis passing through its centre of mass and perpendicular to its length



AB - uniform rod, l - length,

M - mass, PQ - Perpendicular axis

Mass per unit length of the rod

$$m = \frac{M}{l} \rightarrow (1)$$

Consider a small element of length dx of the rod at a distance x, from O

Mass of the element $= m \cdot dx$

M.I of the element about the axis PQ

$$= \text{mass} \times (\text{distance})^2$$
$$= m dx \cdot x^2$$
$$= m x^2 dx \rightarrow (2)$$

Integrating, with in the limits

$x = -l/2$, $x = l/2$

$$I = \int_{-l/2}^{l/2} m x^2 dx$$

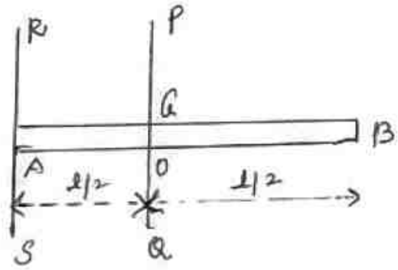
$$= m \int_{-l/2}^{l/2} x^2 dx$$

$$= m \left(\frac{x^3}{3}\right)_{-l/2}^{l/2} = m \left[\frac{(l/2)^3 - (-l/2)^3}{3}\right]$$

$$= m\frac{\left(\frac{l^3}{8} + \frac{l^3}{8}\right)}{3} = \frac{m}{3}\left(\frac{2l^3}{8}\right) = \frac{ml^3}{12} \quad (ml = M)$$

$I = ml \cdot l^2/12$ $\boxed{I = \frac{Ml^2}{12}}$

b) About an axis passing through its one end perpendicular to its length



M.I of the rod about PQ $= \frac{Ml^2}{12}$

By the parallel axis theorem

$$I = \frac{Ml^2}{12} + M(l/2)^2$$

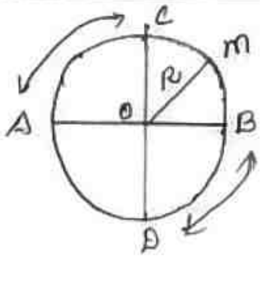
$$I = \frac{Ml^2}{12} + \frac{Ml^2}{4}$$

$$I = \frac{Ml^2 + 3l^2}{12} = \frac{4Ml^2}{12}$$

$$\boxed{I = \frac{Ml^2}{3}}$$

2) Derive an expression for moment of Inertia of a ring or (hoop)

a) About an axis through its centre and perpendicular to its plane



M - mass, R - radius

I - Moment of Inertia

$$I = \Sigma m R^2 \qquad (\because \Sigma m = M)$$

$$\boxed{I = MR^2}$$

(12)

b) About a diameter

$I_z = I_x + I_y \longrightarrow (3)$

$I_z = MR^2$

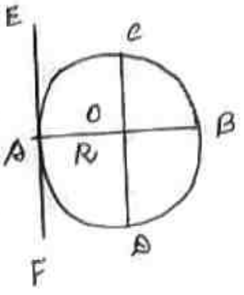
$I_x = I_y = I$

$MR^2 = I + I$

$2I = MR^2$

$$\boxed{I = \frac{MR^2}{2}} \longrightarrow (4)$$

c) About a tangent in the plane of the ring



By parallel axis theorem, M.I about EF

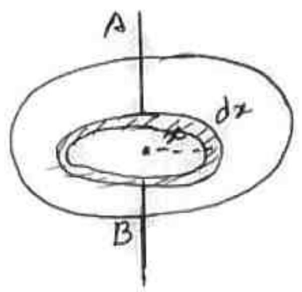
$I = \frac{MR^2}{2} + MR^2$

$I = \frac{MR^2 + 2MR^2}{2}$

$$\boxed{I = \frac{3}{2} MR^2}$$

③ Moment of Inertia of a thin circular disc

a) About an axis through its centre and perpendicular to its plane



m - mass

R - radius

AB - Passing through its Centre O, ⊥r to its plane

Mass per unit area of the disc

$= \frac{M}{\text{Area of the disc}}$

$= \frac{M}{\pi R^2} \longrightarrow (1)$

Area of the strip $= 2\pi x\, dx$

Mass of the strip $= \frac{M}{\pi R^2} 2\pi x\, dx$

$= \frac{2M}{R^2} x\, dx \longrightarrow (2)$

M.I about the axis AB

$= \frac{2M}{R^2} x^3 dx \longrightarrow (3)$

Integrating eqn(3), limits $x = 0$, $x = R$

$I = \int_0^R \frac{2M}{R^2} x^3 dx \longrightarrow (4)$

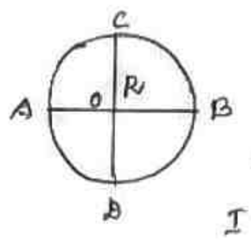
$= \frac{2M}{R^2} \int_0^R x^3 dx$

$= \frac{2M}{R^2} \left[\frac{x^4}{4}\right]_0^R$

$= \frac{2M}{R^2} \frac{R^4}{4}$

$$\boxed{I = \frac{MR^2}{2}} \longrightarrow (5)$$

b) About a diameter



AB, CD - two ⊥r diameters

AB, CD - Equal MI

$I = \frac{MR^2}{2} \longrightarrow (6)$

By theorem of ⊥r axes

$I_z = I_x + I_y \longrightarrow (7)$

$I_z = \frac{MR^2}{2}$, $I_x = I_y = I$

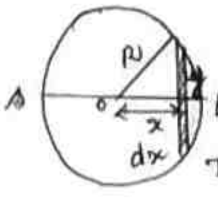
$\frac{MR^2}{2} = I + I$, $MR^2 = 2I$

$I = \frac{MR^2}{2 \times 2}$

$$\boxed{I = \frac{MR^2}{4}} \longrightarrow (8)$$

(4) Moment of Inertia of solid sphere

a) About a diameter



M - mass, R - radius, O - Centre, dx - thickness

The radius of this disc is given by $y^2 = (R^2 - x^2)$

Area of disc $= \pi y^2 \qquad (R^2 = x^2 + y^2)$

$= \pi(R^2 - x^2) \qquad (\because y = R^2 - x^2)$

Volume of the disc = Area × thickness

$= \pi(R^2 - x^2)\,dx \quad \rightarrow (2)$

Mass of the elemental disc $= \dfrac{3M}{4\pi R} \times \pi (R^2 - x^2)\,dx$

$= \dfrac{3M}{4R^3}(R^2 - x^2)\,dx$

M.I of disc about the axis AB

$= \dfrac{\text{Mass} \times (\text{Radius})^2}{2}$

$= \dfrac{3M}{4R^3}(R^2 - x^2)\,dx \times y^2/2$

$= \dfrac{3M}{4R^3}(R^2 - x^2)\,dx \times \dfrac{R^2 - x^2}{2}$

$= \dfrac{3M}{8R^3}(R^2 - x^2)^2\,dx \quad \rightarrow (4)$

x varying from $-R$ to R $\qquad (\because y^2 = R^2 - x^2)$

Moment of Inertia of a solid sphere about a diameter

$$I = \int_{-R}^{R} \frac{3M}{8R^3}(R^2 - x^2)^2\,dx$$

$$= 2\int_{0}^{R} \frac{3M}{8R^3}(R^2 - x^2)^2\,dx$$

$$I = \frac{3M}{4R^3}\int_0^R (R^4 - 2R^2x^2 + x^4)\,dx \quad \rightarrow (5)$$

$$= \frac{3M}{4R^3}\left[R^4x - 2R^2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^R$$

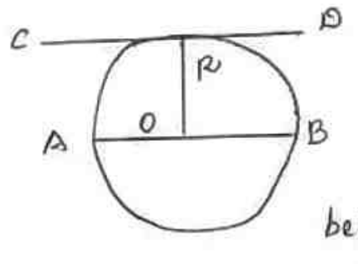
$$= \frac{3M}{4R^2}\left[R^5 - \frac{2R^5}{3} + \frac{R^5}{5}\right]$$

$$= \frac{3M}{4R^2}\left[\frac{15R^5 - 10R^5 + 3R^5}{15}\right]$$

$$= \frac{3M}{4R^3} \times \frac{8R^5}{15}$$

$$\boxed{I = \frac{2}{5}MR^2} \quad \longrightarrow (6)$$

b) About a tangent



R - distance between tangent and diameter

By Parallel axis theorem, about the tangent CD

$I = M\text{-}I$

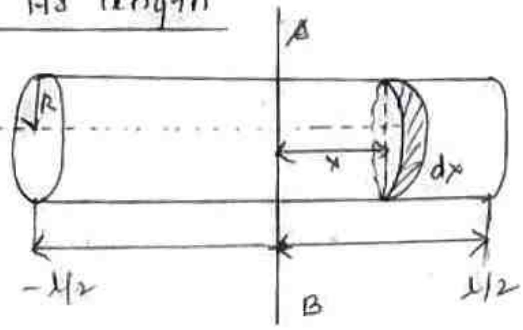
$I = 2/5\,MR^2 + MR^2$

$I = \dfrac{2MR^2 + 5MR^2}{5}$

$$\boxed{I = 7/5\,MR^2}$$

(9) Moment of Inertia of a Solid cylinder

a) About an axis passing through the centre and perpendicular to its length



M - mass, l - length, r - radius

Mass per unit length of cylinder

$$m = \frac{M}{l} \longrightarrow (1)$$

dx - thickness, x - distance from the axis AB.

Mass of the disc $= m\,dx$

Moment of Inertia of the disc about its own diameter

$$= \frac{\text{Mass} \times (\text{Radius})^2}{4} = \frac{m\,dx\,R^2}{4}$$

$$= \frac{mR^2 dx}{4}$$

using parallel axis theorem

$$= \frac{mR^2 dx}{4} + m\,dx\,x^2$$

Integrating above eqn $x = -l/2$, $x = l/2$

M.I of the cylinder about AB

$$I = \int_{-l/2}^{l/2} \left(\frac{mR^2 dx}{4} + m x^2 dx \right)$$

$$= \int_{-l/2}^{l/2} \frac{mR^2 dx}{4} + \int_{-l/2}^{l/2} m x^2 dx$$

$$= 2\int_0^{l/2} \frac{mR^2 dx}{4} + 2\int_0^{l/2} m x^2 dx$$

$$I = \frac{mR^2}{2}(x)_0^{l/2} + 2m\left(\frac{x^2}{3}\right)_0^{l/2} \longrightarrow (6)$$

$$= \frac{mR^2}{2} \times l/2 + 2m \times \frac{l^3}{3 \times 8}$$

$$= \frac{M}{l} \times \frac{R^2}{2} \times l/2 + \frac{2M}{l} \times \frac{l^3}{3 \times 8} \quad \left(\because m = \frac{M}{l}\right)$$

$$I = \frac{MR^2}{4} + \frac{Ml^2}{12}$$

$$\boxed{I = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right)} \longrightarrow (7)$$

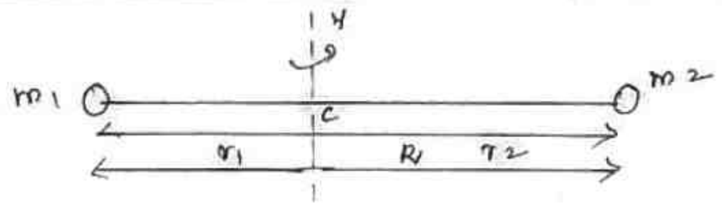
b) About the axis of the cylinder

M.I of a disc about an axis passing through its centre and perpendicular to its plane $= mR^2/2 \rightarrow (8)$

M.I of a solid cylinder $= \Sigma mR^2/2$

$$\boxed{I = \frac{MR^2}{2}} \longrightarrow (9)$$

(10) Discuss the moment of Inertia of a diatomic molecule



- Consider two masses m_1, m_2 seperated by a distance R
- C - Centre of mass, r_1 & r_2 - distances of two atoms

(15)

$r_1 + r_2 = R \rightarrow (1)$

$m_1 r_1 = m_2 r_2 \rightarrow (2)$

From eqn (1)

$r_1 = R - r_2 \rightarrow (3)$

From eqn (2)

$r_2 = \frac{m_1 r_1}{m_2} \rightarrow (4)$

eqn (3) be

$r_1 = R - \frac{m_1 r_1}{m_2}$

$R = r_1 + \frac{m_1 r_1}{m_2} = r_1\left[1 + \frac{m_1}{m_2}\right] \rightarrow (5)$

(or)

$$\boxed{r_1 = \frac{R}{\left(1 + m_1/m_2\right)}} \longrightarrow (6)$$

$I = m_1 r_1^2 + m_2 r_2^2 \rightarrow (7)$

$I = m_1 r_1 . r_1 + m_1 r_1 . r_2$ from eqn (2)

$I = m_1 r_1 (r_1 + r_2)$

by using eqn (1)

$$\boxed{I = m_1 r_1 R}$$

Substituting eqn (6) in (8)

$I = m_1 R \left[\frac{R}{\left(1 + \frac{m_1}{m_2}\right)}\right]$

$I = \frac{m_1 R^2}{\left(1 + \frac{m_1}{m_2}\right)} = \frac{m_1 R^2}{\frac{m_2 + m_1}{m_2}} = \frac{m_1 m_2 R^2}{m_2 + m_1}$ (or)

$I = \left(\frac{m_1 m_2}{m_1 + m_2}\right) R^2$

$$\boxed{I = \mu R^2}$$

$\mu = \frac{m_1 m_2}{m_1 + m_2}$ is called reduced mass

For radius of gyration, $K = R$

$$\boxed{I = \mu K^2}$$

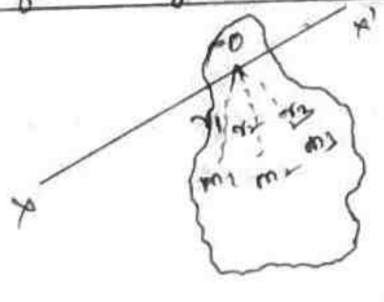
(11) Derive an expression for Angular momentum of a rigid body

Definition

Angular momentum of a particle is defined as its moment of linear momentum it is given by the product of linear momentum and perpendicular distance of its line of action from the axis of rotation. It is denoted by $\vec{L}$

$$\boxed{\vec{L} = \vec{r} \times \vec{P}}$$

Expression for Angular momentum of a rigid body



- Rigid body rotating about a fixed axis XOX'
- m_1, m_2, m_3 - masses
  r_1, r_2, r_3 - distances
  ω - angular velocity

Angular momentum = linear momentum × distance

$= mv \times r$

$= mr\omega \times r$

$= mr^2\omega$ $[v = r\omega]$

Angular momentum of the first particle $= m_1 r_1^2 \omega$

Angular momentum of the second Particle $= m_2 r_2^2 \omega$

Angular momentum of the second Particle $= m_3 r_3^2 \omega$

Angular momentum of the rigid body $\Big\} = m_1 r_1^2 \omega + m_2 r_2^2 \omega + m_3 r_3^2 \omega$

$= \omega (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2)$

$= \omega \sum m r^2$

$I = \sum m r^2$

Angular momentum of the rigid body $= \omega I$

$$\boxed{L = I\omega}$$

(12) Describe Principle, Construction and working of gyroscope, mention its application in various fields.

Definition: A gyroscope is a device consisting of a wheel or disc that spins rapidly about an axis that is also free to change direction.

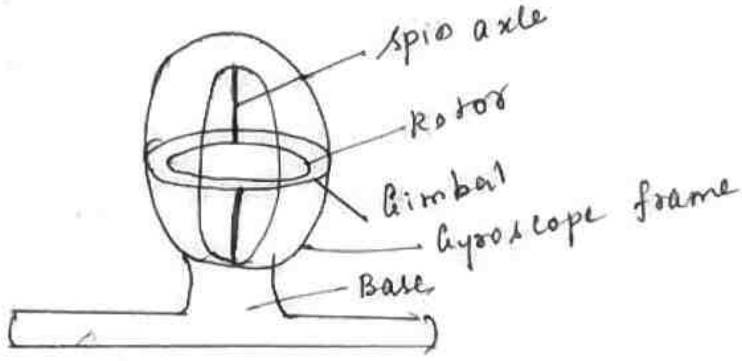
Principle: Based on conservation of angular momentum

Properties: Two basic Properties

i) Rigidity: The axis of rotation of the gyrowheel tends to remain in a fixed direction in space if no force is applied to it.

ii) Precession: The axis of rotation has a tendency to turn at a right angle to the direction of an applied force.

Design of Gyroscope



- Rotor fixed on the supporting rings known as gimbals.
- In central rotor, frictionless bearings present in the gimbals
- Axle of the spinning wheel
- Maintains high speed rotation axis at the central rotor.
- Rotor has three degrees of rotational freedom.

Working

- Gimbals support the weight of the gyroscope.
- Cause no torques.
- If axle is fixed direction, the angular momentum of the gyroscope points along the axle.
- Gyroscope used as navigational device on ships, aeroplanes and spacecraft
- Need to conserve angular momentum
- Gyroscope undergoes a characteristic type of motion called Precession.

Applications

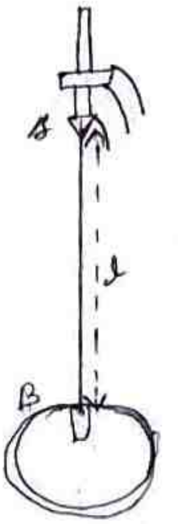
- used as stabilizers in ships, boats and aeroplanes.
- Automatic steering systems, used in airplanes and missiles.
- In gyrocompass, a directional instrument used on ships.

(18) Derive an expression for time period of torsion pendulum. Explain how it is used to find rigidity modulus of a wire.

Definition

A circular metallic disc suspended using a thin wire that executes torsional oscillation is called torsional pendulum

Description



- Upper end fixed
- lower end connected to the centre of a heavy circular disc

Expression for the period of oscillation of a Torsion pendulum

- When disc is rotated by applying twist, wire twisted through an angle θ.

The restoring couple in the wire $= C\theta \longrightarrow (1)$

C - couple per unit twist

Applied couple $= I \dfrac{d^2\theta}{dt^2}$

At equilibrium

applied couple = restoring couple

$$I\frac{d^2\theta}{dt^2} = -C\theta \longrightarrow (2)$$

18

Negative sign indicates the restoring couple is opposite to applied couple

$$\frac{d^2\theta}{dt^2} = -\frac{C}{I}\theta \longrightarrow (3)$$

The time period of oscillation

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

$$= 2\pi\sqrt{\frac{\theta}{C/I \times \theta}}$$

$$T = 2\pi\sqrt{I/C} \longrightarrow (4)$$

Uses of Torsional pendulum

i) Rigidity modulus of the wire
ii) Moment of inertia of the disc
iii) M.I of an Irregular body.

Determination of Rigidity Modulus of the wire

$$T = 2\pi\sqrt{I/C} \longrightarrow (1)$$

- Circular disc suspended by a thin wire
- Top end of the wire is fixed in a vertical support.
- Disc is rotated, executes torsional oscillations.
- Time taken for 20 oscillations noted.
- Experiment repeated, mean time period is determined.

The time period of oscillation

$$T = 2\pi\sqrt{I/C} \longrightarrow (2)$$

squaring on both sides

$$T^2 = 4\pi^2\left(\sqrt{I/C}\right)^2 \longrightarrow (3)$$

$$T^2 = \frac{4\pi^2 I}{C} \longrightarrow (4)$$

$C = \frac{\pi n r^4}{2l}$, substitute C in eqn(4)

$$T^2 = \frac{4\pi^2 I}{\frac{\pi n r^4}{2l}} = \frac{2l \times 4\pi^2 I}{\pi n r^4} \longrightarrow (5)$$

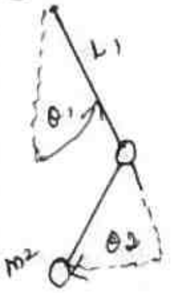
rearrange eqn (5)

$$\boxed{n = \frac{8\pi I}{r^4}\left(l/T^2\right)}$$ - Rigidity Modulus of the wire

$$I = MR^2/2$$

(14) **Write notes on double pendulum**

A system in which a pendulum is attached to the end of another pendulum known as double pendulum



- x - horizontal Position
- y - Vertical Position
- θ - angle of Pendulum
- L - Length of rod
- Let position of pendulum 1 be (x_1, y_1), Pendulum 2 (x_2, y_2)

$$x_1 = L_1 \sin\theta_1$$

$$y_1 = -L_1 \cos\theta_1$$

$$x_2 = x_1 + L_2 \sin\theta_2$$

$$y_2 = y_1 - L_2 \cos\theta_2$$

The velocity is the derivative with respect to time of the position.

$$\frac{dx_1}{dt} = \frac{d\theta_1}{dt} L_1 \cos\theta_1$$

$$\dot{x}_1 = \dot{\theta}_1 L_1 \cos\theta_1$$

$$\dot{y}_1 = \dot{\theta}_1 L_1 \sin\theta_1$$

$$\dot{x}_2 = \dot{x}_1 + \dot{\theta}_2 L_2 \cos\theta_2$$

$$\dot{y}_2 = \dot{y}_1 + \dot{\theta}_2 L_2 \sin\theta_2$$

The acceleration is the second derivative

$$\ddot{x}_1 = -\dot{\theta}_1^2 L_1 \sin\theta_1 + \ddot{\theta}_1 L_1 \cos\theta_1$$

$$\ddot{y}_1 = \dot{\theta}_1^2 L_1 \cos\theta_1 + \ddot{\theta}_1 L_1 \sin\theta_1$$

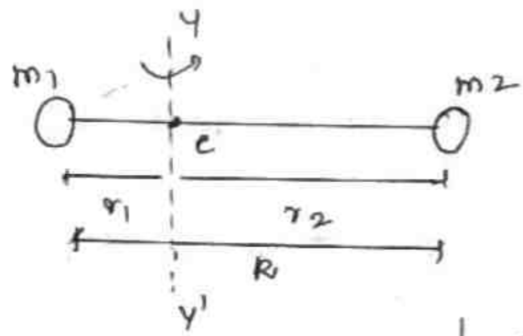
$$\ddot{x}_2 = \ddot{x}_1 - \dot{\theta}_2^2 L_2 \sin\theta_2 + \ddot{\theta}^2 L_2 \cos\theta_2$$

$$\ddot{y}_2 = \ddot{y}_1 \dot{\theta}_2^2 L_2 \cos\theta_2 + \ddot{\theta}^2 L_2 \sin\theta_2$$

Uses of double pendulum :

- Used in education, research and applications
- Used to study chaos both experimentally and numerically

(15) Discuss the rotational energy states of a rigid diatomic molecule



- Consider two masses m_1, m_2
- r_1, r_2 - Distance
- YY' - axis of rotation
- Arrangement called rigid rotor
- R - bond length between two atoms. $(R = r_1 + r_2)$

K.E is given as

$$E = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2$$

$$E = \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2)\omega^2 \longrightarrow (1)$$

$$E = \frac{1}{2} I \omega^2$$

M.g $I = m_1 r_1^2 + m_2 r_2^2$

$$E = \frac{1}{2} I \omega^2 \longrightarrow (2)$$

Eqn (2) rewritten as

$$E = \frac{1}{2I} \cdot I^2 \omega^2 \longrightarrow (3)$$

$$I\omega = L$$

Eqn (3) becomes

$$E = \frac{L^2}{2I} \longrightarrow (4)$$

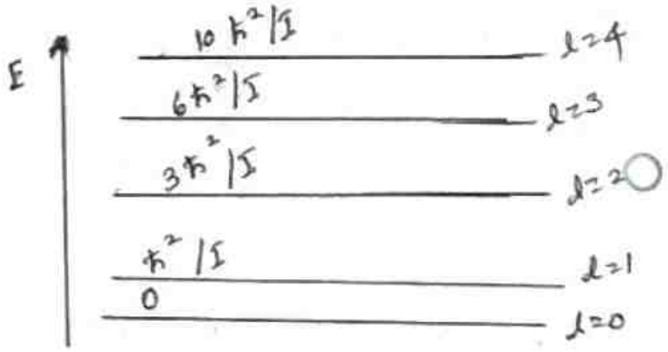
$$L^2 = l(l+1)\hbar^2, \quad l = 0, 1, 2 \ldots (5)$$

l - rotational quantum number

l varies interms of integer values

$$E_l = \frac{l(l+1)\hbar^2}{2I} \longrightarrow (6)$$

$\hbar = h/2\pi$, h - Planck's Constant



Levels are not equally spaced.

Unit - II

Electromagnetic waves

The Maxwell's eqns - wave eqn : plane electromagnetic waves in a vacuum, conditions on the wave field - Properties of electromagnetic waves ; speed, amplitude, Phase orientation and waves in matter - Polarization - Producing electromagnetic waves - Energy and momentum in EM waves ; Intensity waves from localized sources momentum and radiation Pressure - cell phone reception . Reflection and transmission of electromagnetic waves from a non-conducting medium-vacuum interface for normal incidence.

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① **Derive Maxwell's eqns in differential and Integral form**

Maxwell's eqn - I (from Gauss law in electrostatics)

Integral form

Gauss's law in electrostatics state that the total electric flux through any closed surface is equal to the charge enclosed by it

According to Gauss law

$$\oint\!\!\oint_S \vec{E}\cdot\vec{ds} = q/\varepsilon \rightarrow (1)$$

$$\oint\!\!\oint_S \varepsilon\vec{E}\cdot\vec{ds} = q \qquad (\vec{D} = \varepsilon\vec{E})$$

$$\oint\!\!\oint \vec{D}\cdot\vec{ds} = q \longrightarrow (2)$$

Total charge inside the closed surface is

$$q = \iiint_V \rho\, dv \longrightarrow (3)$$

Substituting eqn(3) in (2)

$$\oint\!\!\oint \vec{D}\cdot\vec{ds} = \iiint_V \rho\, dv \rightarrow (4)$$

Eqn (4) is the Maxwell's eqn in integral form from Gauss law in electrostatics.

Applying Gauss's divergence theorem to LHS of eqn (4)

$$\oint\!\!\oint_S \vec{D}\cdot\vec{ds} = \iiint_V \vec{\nabla}\cdot\vec{D}\, dv \rightarrow (5)$$

on substituting eqn(5) in eqn(4)

$$\iiint_V \vec{\nabla}\cdot\vec{D}\, dv = \iiint_V \rho\, dv \rightarrow (6)$$

(or) $\vec{\nabla}\cdot\vec{D} = \rho$

$$\vec{\nabla}\,\varepsilon_0\vec{E} = \rho \qquad (D = \varepsilon_0\vec{E})$$

$$\varepsilon_0\vec{\nabla}\cdot\vec{E} = \rho$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0}$$

This is Maxwell's eqn from Gauss' law in electrostatics in differential form.

Maxwell's eqn II [from Gauss's law in magnetostatics]

Integral form

Total magnetic flux through any closed surface in a magnetic field is zero.

$$\oint \vec{B}\cdot\vec{ds} = 0 \longrightarrow (1)$$

This is maxwell's eqn in integral form from Gauss's law in magnetostatics.

L.H.S of eqn (1)

$$\oint \vec{B}\cdot\vec{ds} = \iiint_V \vec{\nabla}\cdot\vec{B}\,dv \longrightarrow (2)$$

Substituting eqn (2) in (1)

$$\iiint_V \vec{\nabla}\cdot\vec{B}\,dv = 0 \longrightarrow (3)$$

(or) $$\boxed{\vec{\nabla}\cdot\vec{B} = 0} \longrightarrow (4)$$

This is Maxwell's eqn in differential form from Gauss's law in magnetostatics.

Maxwell's eqn (III) (From Faraday's law)

Magnetic flux through a small area $ds = \vec{B}\cdot\vec{ds} \longrightarrow (1)$

Total magnetic flux linked with the circuit $\phi_B = \oint_S \vec{B}\cdot\vec{ds}$

$$\longrightarrow (2)$$

Faraday's law states that the induced emf e is the rate of change of magnetic flux ϕ_B

$$e = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\left[\oint \vec{B}\cdot\vec{ds}\right] \longrightarrow (3)$$

$$= \oint \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds}$$

$\vec{E}$ - Electric field strength

$\vec{E} = \frac{dv}{dl}$, $dv = \vec{E}\cdot\vec{dl}$

$V = \int dv = \int \vec{E}\cdot\vec{dl}$

$V = e = \int \vec{E}\cdot\vec{dl}$

$$e = \oint \vec{E}\cdot\vec{dl} \longrightarrow (4)$$

Equating eqn (3) in eqn (4)

$$\oint_C \vec{E}\cdot\vec{dl} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds} \longrightarrow (5)$$

This is Maxwell's eqn in integral form from Faraday's law of electromagnetic induction.

Applying stoke's theorem to L.H.S. eqn (5)

$$\oint_C \vec{E}\cdot\vec{dl} = \iint_S (\vec{\nabla}\times\vec{E})\cdot\vec{ds} \longrightarrow (6)$$

Substituting eqn (6) in (5)

$$\iint_S (\vec{\nabla}\times\vec{E})\,\vec{ds} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds} \longrightarrow (7)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow (8)$$

Eqn (8) represents Maxwell's eqn from faraday's law of electro magnetic induction in differential form.

Statement : The electromotive force around a closed path is equal to the rate of magnetic displacement through the closed path.

Maxwell's eqn (IV)
from Ampere's circuital law

Ampere's law states that the line integral of magnetic field intensity H on any closed path is equal to the current (I) enclosed by that path.

$$\oint \vec{H}\cdot d\vec{l} = I \longrightarrow (1)$$

$J = I/A \quad , \quad I = JA$

$$I = J\iint_S ds \qquad (\because A = \iint_S ds)$$

$$I = \iint_S \vec{J}\cdot ds \longrightarrow (2)$$

Substituting eqn (2) in (1)

$$\oint \vec{H}\cdot d\vec{l} = \iint_S \vec{J}\cdot d\vec{s} \longrightarrow (3)$$

Ampere's law is modified by introducing displacement current density

$$\oint \vec{H}\cdot d\vec{l} = \iint_S (\vec{J}_c + \vec{J}_D)\, d\vec{s} \longrightarrow (4)$$

Substituting $\vec{J}_c = \sigma E$,

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t}$$

$$\oint \vec{H}\cdot d\vec{l} = \iint_S \left(\sigma\vec{E} + \frac{\partial \vec{D}}{\partial E}\right) ds$$

$$\oint \vec{H}\cdot d\vec{l} = \iint_S \left(\sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t}\right) ds \longrightarrow (5)$$

$(J = J_c)$

$$\oint \vec{H}\cdot d\vec{l} = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) ds \longrightarrow (6)$$

$(J = \sigma E, \; D = \varepsilon\vec{E})$

This is Maxwell's eqn in integral form from Ampere's circuital law

Applying stoke's theorem to L.H.S of eqn (6)

$$\oint \vec{H}\cdot d\vec{l} = \iint_S (\vec{\nabla}\times\vec{H})\, ds \longrightarrow (7)$$

Substituting eqn (7) in (6)

$$\iint_S (\vec{\nabla}\times\vec{H})\, ds = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) ds \longrightarrow (8)$$

$$(\vec{\nabla}\times\vec{H}) = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow (9)$$

$$\vec{\nabla}\times\vec{H} = \sigma E + \varepsilon\frac{\partial \vec{E}}{\partial t} \longrightarrow (10)$$

Eqns (9) & (10) are Maxwell's eqns in differential form from Ampere's circuital law.

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② Deduce Maxwell's eqns for free space

Four Maxwell's eqn in differential form

$$\vec{\nabla}\cdot\vec{D} = \rho \longrightarrow (1) \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \longrightarrow (2) \qquad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \longrightarrow (4)$$

③

Maxwell eqns reduce to

$\vec{\nabla}\cdot\vec{D} = 0 \rightarrow (5)$ $(\because \rho = 0)$

$\vec{\nabla}\cdot\vec{B} = 0 \rightarrow (6)$

$\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \rightarrow (7)$

$\vec{\nabla}\times\vec{H} = \dfrac{\partial D}{\partial t} \rightarrow (8)$ $(\because J = 0)$

Maxwell's eqns in Conducting Media

$\vec{J} = \sigma\vec{E}$, σ - Electrical Conductivity

$\vec{B} = \mu\vec{H}$, μ - Permeability

$\vec{D} = \varepsilon\vec{E}$, ε - Permittivity

General Maxwell eqns reduce to

$\nabla\cdot\vec{D} = \rho \rightarrow (1)$

$\nabla\cdot\vec{B} = 0 \rightarrow (2)$

$\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \rightarrow (3)$

$\vec{\nabla}\times\vec{H} = \vec{J} + \dfrac{\partial\vec{D}}{\partial t} \rightarrow (4)$

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3) Explain the Characteristics of Maxwell's Eqn

i) Maxwell's first Eqn $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$

- Explains Gauss's law in electrostatics
- Time independent or steady state eqn.
- The flux of the lines of electric force depends upon charge density.
- charge acts as a source.

ii) Maxwell's second Eqn $\vec{\nabla}\cdot\vec{B} = 0$

- Explains Gauss's law in Magnetostatics.
- Time independent eqn
- No source

iii) Maxwell's third Eqn

$\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

- It explains Faraday's law and lenz's law
- Time independent eqn
- $\vec{E}$ is generated by the time variation of $\vec{B}$

iv) Maxwell's fourth Eqn

$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \dfrac{\partial\vec{D}}{\partial t}\right)$

- Relation with magnetic field vector $\vec{B}$, with displacement vector $\vec{D}$ and the current density.
- Time dependent eqn
- Explains Ampere's circuital law.

(4) Discuss the plane wave eqn

Definition :

If a wave is confined to a particular axis with equal magnitudes of electric and magnetic field vectors then that wave is called plane wave

Plane Electromagnetic wave Eqn in vacuum

Maxwell's eqns in general form

$\vec{\nabla}\cdot\vec{D} = \rho \rightarrow (1)$ | $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$\vec{\nabla}\cdot\vec{B} = 0$ | $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \rightarrow (3)$

$\vec{D} = \varepsilon_0\vec{E}$, $\vec{B} = \mu_0\vec{H} \rightarrow (4)$

Conductivity $\sigma = 0$

$\vec{J} = 0 \quad (\therefore \vec{J} = \sigma\vec{E},\ \sigma = 0)$

No charge Present in the Vacuum, $\rho = 0$ eqn (1) reduces to

$\vec{\nabla}\cdot\vec{D} = 0$ (or)

$\vec{\nabla}\cdot\varepsilon_0\vec{E} = 0 \quad (\therefore \vec{D} = \varepsilon_0\vec{E})$

$\varepsilon_0\vec{\nabla}\cdot\vec{E} = 0$

$\boxed{\vec{\nabla}\cdot\vec{E} = 0}$

Wave eqn for electric field vector ($\vec{E}$)

Taking Curl on both sides of eqn (3)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$= \frac{\partial}{\partial t}(\vec{\nabla}\times\mu_0\vec{H})$$

$(\therefore \vec{B} = \mu_0\vec{H})$

(or) $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$

Now from vector calculus identity

$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} \rightarrow (7)$

from eqn (5)

$\vec{\nabla}\cdot\vec{E} = 0$, substituting this in eqn (7)

$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2\vec{E} \rightarrow (8)$

Substituting eqn (8) in eqn (6)

$$-\nabla^2 E = -\mu_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

Substituting for $\vec{\nabla}\times\vec{H}$ from eqn (4)

$$-\nabla^2 E = -\mu_0\frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) \rightarrow (9)$$

(or)

$$-\nabla^2\vec{E} = -\mu_0\frac{\partial}{\partial t}\left[\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right]$$

$(\therefore \vec{J} = 0 ,\ \vec{D} = \varepsilon_0\vec{E})$

$$\boxed{\nabla^2 E = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}} \rightarrow (10)$$

Eqn (10) is the general electro magnetic wave eqn.

Wave eqn for magnetic field Vector ($\vec{B}$)

Taking curl on both sides of the eqn (4)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\times\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) \text{ (or)}$$

$(\therefore \vec{J} = 0$ and $\vec{D} = \varepsilon_0\vec{E})$

From vector calculus identity

$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \nabla^2\vec{H}$

(5)

From eqn (2)

$\vec{\nabla}\cdot\vec{B} = 0$

$\mu_0 (\vec{\nabla}\cdot\vec{H}) = 0$ or $(\vec{\nabla}\cdot\vec{H}) = 0$

Substituting eqn (13) in $(\vec{B} = \mu_0 \vec{H})$ eqn (12)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = -\nabla^2\vec{H} \longrightarrow (14)$$

Using eqn (14) and eqn (11)

$$-\nabla^2\vec{H} = \varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) \longrightarrow (15)$$

Substituting the eqn(3) in eqn (15)

$$-\nabla^2\vec{H} = \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$-\nabla^2\vec{H} = -\varepsilon_0 \frac{\partial^2}{\partial t^2}(\mu_0\vec{H}) \quad \left(\because \vec{B} = \mu_0\vec{H}\right)$$

$$\boxed{\nabla^2\vec{H} = \mu_0\varepsilon_0 \frac{\partial^2\vec{H}}{\partial t^2}} \longrightarrow (16)$$

This general electromagnetic wave eqn interms of $\vec{H}$ for free space.

The electromagnetic wave eqn for $\vec{E}$ and $\vec{H}$ is written as

$$\nabla^2 E - \mu_0\varepsilon_0 \frac{\partial^2\vec{E}}{\partial t^2} = 0 \longrightarrow (17)$$

$$\nabla^2\vec{H} - \mu_0\varepsilon_0 \frac{\partial^2\vec{H}}{\partial t^2} = 0 \longrightarrow (18)$$

In one dimension say along x-axis, the wave eqns are given by the x-components of the above expression.

$$\frac{\partial^2 E_x}{\partial x^2} - \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0 \longrightarrow (19)$$

$$\frac{\partial^2 H_x}{\partial x^2} - \mu_0\varepsilon_0 \frac{\partial^2 H_x}{\partial t^2} = 0 \longrightarrow (20)$$

Speed (velocity) of EM wave in Vaccum

Comparing eqn (19) & (20)

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t} = 0 \longrightarrow (21)$$

y - instantaneous displacement

c - velocity of wave

The velocity (speed) of the electromagnetic wave is given by

$$1/c^2 = \mu_0\varepsilon_0, \quad c^2 = 1/\mu_0\varepsilon_0$$

Magnitude of velocity is called speed

$$c = 1/\sqrt{\mu_0\varepsilon_0} \longrightarrow (22)$$

For vacuum or free space

$\mu_0 = 4\pi\times10^{-7}\,Hm^{-1}$ (henry per metre)

$\varepsilon_0 = 8.842\times10^{-12}\,Fm^{-1}$ (farad per metre)

Substituting these values in eqn (22)

$$\boxed{c = 2.998\times10^{8}\,ms^{-1}}$$

Wave eqns for Plane Polarised EM wave in free space and their Solution

The EM wave eqns, for $\vec{E}$ and $\vec{H}$ in free space are given by

$$\nabla^2 E - \mu_0\varepsilon_0 \frac{\partial^2\vec{E}}{\partial t^2} = 0 \longrightarrow (1)$$

$$\nabla^2 H - \mu_0\varepsilon_0 \frac{\partial^2 H}{\partial t^2} = 0 \longrightarrow (2)$$

Conditions on the wave field

If the Plane Polarized waves is Propagating along x-axis having electric Vector along the y-axis

$E_y \neq 0,\ E_z = E_x = 0$

For magnetic field vector

$H_z \neq 0,\ H_y = H_x = 0$

The wave eqns for plane electromagnetic wave reduce to

$$\nabla^2 E_y - \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} = 0 \longrightarrow (3)$$

$$\nabla^2 H_z - \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2} = 0 \longrightarrow (4)$$

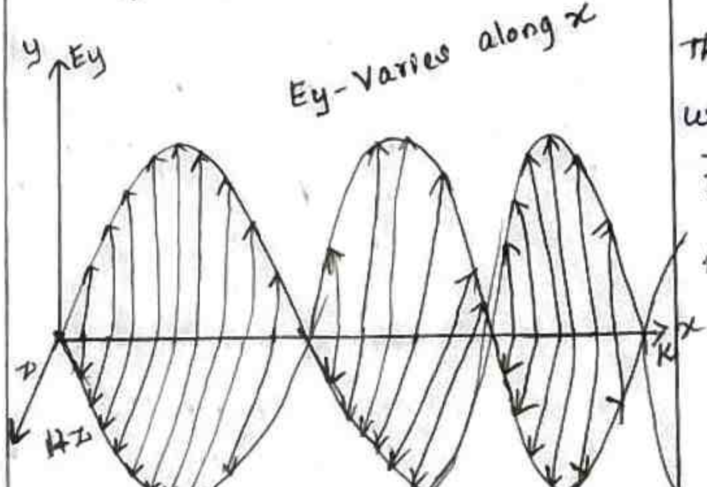
$$\nabla^2 E_y = \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \longrightarrow (5)$$

$$\nabla^2 H_z = \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} \longrightarrow (6)$$

$$\frac{\partial^2 E_y}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 E_y}{\partial z^2} = 0$$

Similarly

$$\frac{\partial^2 H_z}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 H_z}{\partial z^2} = 0$$



$$\nabla^2 E_y = \frac{\partial^2 E_y}{\partial x^2}$$

$$\nabla^2 H_z = \frac{\partial^2 H_z}{\partial x^2}$$

substituting eqn(7) in eqn(3) and eqn(8) in eqn(4)

$$\frac{\partial^2 E_y}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} = 0 \longrightarrow (9)$$

$$\frac{\partial^2 H_z}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2} = 0 \longrightarrow (10)$$

Solutions of the Plane wave eqns

The plane wave eqns for electric field and magnetic field are given by

$$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = 0$$

$$\frac{\partial^2 H_z}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 H_z}{\partial t^2} = 0 \qquad (\because \mu_0 \varepsilon_0 = 1/c^2)$$

c – Speed of EM wave

The solutions of the above wave eqns of progressive wave are given by

$$E_y = E_0 \cos(\omega t - kx) \longrightarrow (11)$$

$$H_z = H_0 \cos(\omega t - kx) \longrightarrow (12)$$

The general solution of the wave eqn is written as

$$\vec{E_y} = E_0 e^{i(\omega t - kx)} = E_0 e^{ik(ct - x)}$$

$$H_z = H_0 e^{i(\omega t - kx)} = H_0 \, e^{ik(ct - x)} \qquad (\because c = \nu\lambda)$$

where

$$k = \frac{2\pi}{\lambda}, \quad \omega = 2\pi\nu = \frac{2\pi c}{\lambda} = kc$$

$c \rightarrow$ wave velocity

(5) Explain phase and orientation of EM wave in matter

Electric and Magnetic fields are same $(\omega t - kx)$. Both fields are in phase with each other.

Relation between electric and magnetic field vectors

For electromagnetic waves in free space

$$\vec{E_y} = E_0 e^{ik(ct-x)} \longrightarrow (1)$$

$$\vec{H_z} = H_0 e^{ik(ct-x)} \longrightarrow (2)$$

The relation between their time and space variations is given from Maxwell's eqn

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (\because \vec{B} = \mu_0 \vec{H})$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad (\because E_x = E_z = 0)$$

(or) $$\frac{\partial \vec{E_y}}{\partial x} = -\mu_0 \frac{\partial \vec{H_z}}{\partial t} \longrightarrow (3)$$

Substituting E_y and H_z from the eqns (1) & (2) in eqn (3)

$$\frac{\partial}{\partial x}\left(E_0 e^{ik(ct-x)}\right) = -\mu_0 \frac{\partial}{\partial t}\left(H_0 e^{ik(ct-x)}\right) \longrightarrow (4)$$

$$-ikE_0 e^{ik(ct-x)} = -\mu_0 (ikc) \cdot H_0 e^{ik(ct-x)}$$

$$E_0 = \mu_0 c H_0 \longrightarrow (5)$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \longrightarrow (6)$$

Substituting eqn (6) in eqn (5)

$$E_0 = \mu_0 \times \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \times H_0 = \sqrt{\mu_0/\varepsilon_0} \cdot H_0$$

$$\sqrt{\mu_0/\varepsilon_0} = \frac{E_0}{H_0} = \frac{E_0 e^{i(\omega t - kx)}}{H_0 e^{i(\omega t - kx)}}$$

$$\frac{\vec{E}}{\vec{H}} = \sqrt{\mu_0/\varepsilon_0} \longrightarrow (7)$$

This is the relation between the electric field vector and magnetic field vector.

$$\frac{\vec{E}}{\vec{H}} = \frac{E_0}{H_0} = \sqrt{\mu_0/\varepsilon_0}$$

The ratio $\frac{E}{H}$ is having the unit of impedance (Resistance) i.e) ohm

The quantity $\sqrt{\mu_0/\varepsilon_0}$ has the dimensions of impedance

$$\sqrt{\mu_0/\varepsilon_0} = \sqrt{\frac{H/m}{F/m}} = \sqrt{\frac{henry/m}{farad/m}}$$

$$\sqrt{\frac{henry}{farad}} = \frac{ohm \times sec}{Joul / Volt} = \sqrt{\frac{ohm \times volt}{Joule/sec}}$$

$$= \sqrt{amp \times \frac{Volt}{ohm}}$$

$$= \sqrt{ohm \times ohm} = ohm.$$

It is known as Intrinsic or characteristic impedance of free space, denoted by Z_0.

$\vec{E}$ is parallel to Y-axis. The vector $(\vec{E} \times \vec{H})$ is known as Poynting Vector.

(6) What is mean by Poynting vector? What is its Significance

The cross product of electric field vector $\vec{E}$ and the magnetic field vector $\vec{H}$ is called Poynting Vector. It is denoted by $\vec{S} = \vec{E} \times \vec{H}$

- A plane Polarized electro magnetic wave is propagating along the x-axis.
- Electric vector is directed along the Y-axis
- Magnetic vector is directed along the Z-axis

$$\vec{S} = \vec{E} \times \vec{H} = \hat{j} E_y \times \hat{k} H_z$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & E_y & 0 \\ 0 & 0 & H_z \end{vmatrix} = \hat{i}\left(E_y H_z\right)$$

. Poynting vector gives the time ratio of flow of electromagnetic wave energy per unit area of the medium.

. The average poynting vector for one complete cycle of electromagnetic wave is given by

$$S_{avg} = \frac{1}{2}\left(\vec{E} \times \vec{H}\right)$$

$$= \frac{1}{2} E_0 \times H_0$$

$$= \frac{E_0}{\sqrt{2}} \times \frac{H_0}{\sqrt{2}} = E_{rms} \cdot H_{rms}$$

$$\left(\because E_{rms} = \frac{E_0}{\sqrt{2}}, \; H_{rms} = \frac{H_0}{\sqrt{2}} \right)$$

Significance

- If there is a varying electric field in vacuum, there is also a varying magnetic field.
- Electric and magnetic fields obey wave eqn.
- The speed of propagation given by $C = 1/\sqrt{\mu_0 \epsilon_0}$ is the same as the measured speed of light.
- Light waves, be identified as electromagnetic waves.

(7) Discuss propagation of electromagnetic wave through a Dielectric medium. (Non Conducting Isotropic Medium)

Maxwell's eqns are

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

In an isotropic dielectric

$\vec{D} = \varepsilon \vec{E}$, $\vec{B} = \mu \vec{H}$, $\vec{J} = \sigma \vec{E} = 0$

and $\rho = 0$

Therefore Maxwell's eqns

$$\vec{\nabla} \cdot \vec{E} = 0 \longrightarrow (1)$$

$$\vec{\nabla} \cdot \vec{H} = 0 \longrightarrow (2)$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial B}{\partial t} \longrightarrow (3)$$

$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial E}{\partial t} \longrightarrow (4)$$

<u>Equation of propagation of Magnetic Vector, H</u>

Taking curl of eqn (4)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \varepsilon \vec{\nabla} \times \left(\frac{\partial \vec{E}}{\partial t}\right)$$

(or)

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{H} - \nabla^2 \vec{H} = \varepsilon \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) \longrightarrow (5)$$

Putting values from the eqns (2) & (3)

$$\nabla^2 H = \mu \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \longrightarrow (6)$$

Equation of propagation of electric vector, E

Taking of eqn (3)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\mu \frac{\partial \vec{H}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} \cdot (\vec{\nabla} \times \vec{H})$$

Putting values from eqns (1) and (4)

$$\nabla^2 E = \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \longrightarrow (7)$$

The eqns (6) and (7) compared with general wave eqn

$$\nabla^2 v = \frac{1}{v^2} \frac{\partial^2 v}{\partial t^2}$$

$v = \frac{1}{\sqrt{\mu \varepsilon}}$, v – speed of wave

$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$, speed of electromagnetic

Refractive index is

$$n = \frac{c}{v} = \sqrt{\frac{\mu \varepsilon}{\mu_0 \varepsilon_0}} = \sqrt{\mu_r \varepsilon_r}$$

In a non-magnetic medium

$\mu_r = 1$

$$\boxed{n = \sqrt{\varepsilon_r}}$$

(8) <u>Discuss electromagnetic wave in conducting Medium (Medium with finite μ, ε and σ)</u>

General Maxwell's eqns

$$\vec{\nabla} \cdot \vec{D} = \rho \longrightarrow (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \longrightarrow (2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow (3)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{B}}{\partial t} \longrightarrow (4)$$

In conducting medium $\sigma \neq 0$

$\rho = 0$

eqn(1) reduces to $\vec{\nabla}\cdot\vec{D} = 0$ (or)

$\vec{\nabla}\cdot(\varepsilon\vec{E}) = 0 = \vec{\nabla}\cdot\vec{E} = 0$ $\rightarrow (5)$

$\vec{D} = \varepsilon\vec{E}$, ε - Permittivity of the medium

Taking the curl on both sides of eqn (3)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) \rightarrow (6)$$

From vector calculus identity

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E})$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} \rightarrow (7)$$

But from eqn (5) $\vec{\nabla}\cdot\vec{E} = 0$

Therefore eqn (7) becomes

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2\vec{E} \rightarrow (8)$$

Also

$$\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}) \rightarrow (9)$$

$$(\because \vec{B} = \mu H)$$

Substituting the eqn (8) and eqn (9) in (6)

$$-\nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

$$\nabla^2 E = \mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}) \text{ (or)} \rightarrow (10)$$

Substituting the value of $\vec{\nabla}\times\vec{H}$ from eqn (4) to eqn (10)

$$\nabla^2 E = \mu\frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) \rightarrow (11)$$

Since $\vec{J} = \sigma\vec{E}$ and $\vec{D} = \varepsilon\vec{E}$

eqn (11) becomes

$$\nabla^2\vec{E} = \mu\frac{\partial}{\partial t}\left[\sigma\vec{E} + \frac{\partial}{\partial t}(\varepsilon\vec{E})\right] \text{ (or)}$$

$$\nabla^2 E = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 E - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = 0$$

This is the general wave eqn for the electric vector in an electromagnetic wave propagating in conducting medium. $\rightarrow (12)$

By taking curl of the eqn (4)

$$\nabla^2 H - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} - \mu\sigma\frac{\partial\vec{H}}{\partial t} = 0 \rightarrow (13)$$

wave eqn for plane polarized EM waves

Consider electromagnetic wave is travelling in the x-direction and the electric vector is directed along the y-axis and the magnetic vector is directed along the z-axis.

$E_y \neq 0$, $E_z = E_x$ and $H_z \neq 0$,

$H_y = H_x = 0$

wave eqns from (12) & (13)

$$\frac{\partial^2 E_y}{\partial x^2} - \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2} - \mu\sigma\frac{\partial E_y}{\partial t^2} = 0 \rightarrow (14)$$

$$\frac{\partial^2 H_z}{\partial x^2} - \mu\varepsilon\frac{\partial^2 H_z}{\partial t^2} - \mu\sigma\frac{\partial H_z}{\partial t} = 0 \rightarrow (15)$$

11

above eqns $\mu \varepsilon = 1/v^2$

v - velocity of electromagnetic wave.

The Product $\mu\sigma$ is called magnetic diffusivity.

Solution of the plane EM wave eqn in conduction medium ($\sigma \neq 0$)

eqn (14) in the function of t and in the form

$$\vec{E}_y = E_0 e^{(i\omega t \pm \gamma x)} \longrightarrow (16)$$

soln of eqn (16)

$$\vec{H}_z = H_0 e^{(i\omega t \pm \gamma x)} \longrightarrow (17)$$

Substituting eqn (16) in eqn (14)

$$\frac{\partial^2}{\partial x^2}\left[E_0 e^{(i\omega t \pm \gamma x)}\right] - \mu\varepsilon \frac{\partial^2}{\partial t^2}\left(E_0 e^{(i\omega t \pm \gamma x)}\right) - \mu\sigma \frac{\partial}{\partial t}\left[E_0 \cdot e^{(i\omega t \pm \gamma x)}\right] = 0$$

$$\gamma^2 E_0 e^{(i\omega t \pm \gamma x)} - \mu\varepsilon (i\omega)^2 E_0 e^{(i\omega t \pm \gamma x)} - \mu\sigma i\omega E_0 e^{(i\omega t \pm \gamma x)} = 0$$

$$\gamma^2 - \mu\varepsilon i^2 \omega^2 - \mu\sigma i\omega = 0$$

$$\gamma^2 + \mu\varepsilon\omega^2 - i\mu\sigma\omega = 0 \quad (i^2 = -1)$$

$$\gamma^2 = i\mu\sigma\omega - \mu\varepsilon\omega^2 \longrightarrow (18)$$

$\mu\varepsilon\omega^2$ can be neglected as as compared to $\mu\sigma\omega$.

From eqn (18)

$$\gamma^2 = i\mu\sigma\omega$$

$$\gamma^2 = \frac{2i\mu\sigma\omega}{2} \quad (\text{or})$$

Taking square root on both sides

$$\gamma = \pm (1+i) \sqrt{\frac{\mu\sigma\omega}{2}} = \pm (1+i) K$$

$$\gamma = (1+i) K \quad (\text{or})$$

$$\gamma = -(1+i) K$$

$K = \sqrt{\frac{\mu\sigma\omega}{2}}$ is a constant

Taking -ve values of γ which gives the wave Propagation in the +ve x direction, substituting in eqn (16).

$$\vec{E}_y = E_0 e^{(i\omega t - (1+i)Kx)}$$

$$\vec{E}_y = E_0 e^{(i\omega t - Kx - iKx)}$$

$$\vec{E}_y = E_0 e^{-Kx} e^{i(\omega t - Kx)}$$

This is a progressive wave having amplitude equal to $E_0 e^{-Kx}$

(9) Determine skin depth in conducting material (or)

Penetration depth

In conducting medium amplitude of the electromagnetic wave decreases exponentially with distance of penetration of the wave

The amplitude of a depth x is denoted by $E_0 x$

$$E_0 x = E_0 e^{-Kx} \longrightarrow (1)$$

$$K = \sqrt{\frac{\mu \sigma \omega}{2}}$$

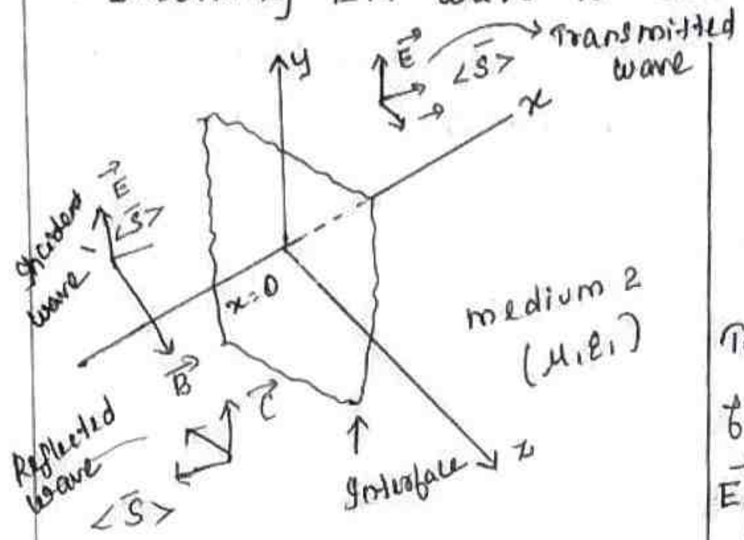
It is defined as the distance inside the conductor from the surface of the conductor at which the amplitude of the field vector is reduced to $1/e$ times its value at the surface.

(10) Discuss the properties of Electromagnetic waves

- Electromagnetic waves are transverse in nature
- Produced by accelerated charges
- EM waves travel with speed of light and doesn't need any medium to propagate.
- EM waves are not deflected by electric or magnetic field.
- Exhibit interference or diffraction and can be polarized.
- EM waves being chargeless
- The energy in an EM wave is equally divided between electric and magnetic field vector.

(11) Discuss propagation EM wave from vaccum to a non-Conducting medium.

- A monochromatic uniform plane wave travels through one medium and enters another medium.
- Incoming EM wave is called the incident wave



$$\vec{E}_i(x,t) = E_0 \cos(\omega t - k_1 x) \rightarrow (1)$$

$$\vec{B}_i(x,t) = \frac{E_0}{v_1} \cos(\omega t - k_1 x) \rightarrow (2)$$

Reflected waves are represented

$$\left(\because B_0 = E_0/v_1\right)$$

$$\vec{E}_R(x,t) = E_1 \cos(\omega t + k_1 x) \rightarrow (3)$$

$$\vec{B}_R(x,t) = \frac{E_1}{v_1}(\cos \omega t + k_1 x) \rightarrow (14)$$

$$\vec{E}_T(x,t) = E_2 \cos(\omega t - k_2 x) \rightarrow (15)$$

$$E_0 = C B_0$$

$$\vec{B}_T(x,t) = \frac{E_2}{V_2} \cos(\omega t - k_2 x) \rightarrow (16)$$

eqn (3) & (4) sign is reversed in the wave number k, along negative x direction

$K_1, K_2 \rightarrow$ wave numbers

$$k_1 = \omega/v_1 \rightarrow (7)$$

$$K_2 = \omega/v_2 \rightarrow (8)$$

v_1, v_2 - Velocity

Total instantaneous electric field $\vec{E}_y$

$$\vec{E}_y(x,t) = E_0 \cos(\omega t - k_1 x) + E_1 \cos(\omega t + k_1 x) \rightarrow (9)$$

$$\vec{E}_y(x,t) = \vec{E}_i(x,t) + \vec{E}_R(x,t) \rightarrow (10)$$

$$\vec{E}_y(x,t) = E_2 \cos(\omega t - k_2 x) \rightarrow (11)$$

At interface $x = 0$

Since the waves are transverse $\vec{E}$, $\vec{B}$ fields tangential to the interface.

$$E_0 \cos(\omega t - k_1 x) + E_1 \cos(\omega t - k_1 x) = E_2 \cos(\omega t - k_2 x)$$

$x = 0$ $\rightarrow (12)$

$$E_0 \cos(\omega t) + E_1 \cos(\omega t) = E_2 \cos(\omega t) \text{ (or)}$$

$$E_0 + E_1 = E_2 \rightarrow (13)$$

At boundary $x=0$,

$$\frac{dE_i}{dx} + \frac{dE_R}{dx} = \frac{dE_T}{dx} \rightarrow (14)$$

$$-E_0 k_1 \sin(\omega t) - E_1 k_1 \sin(\omega t) = E_2 k_2 \sin(\omega t) \rightarrow (15)$$

(or)

$$E_0 k_1 - E_1 k_1 = E_2 k_2$$

(or) $k_1 (E_0 - E_1) = E_2 k_2$

$$E_0 - E_1 = E_2 (k_2/k_1) \rightarrow (16)$$

$k_1 = \omega/v_1$, $k_2 = \omega/v_2$

then eqn (16)

$$E_0 - E_1 = E_2 (v_1/v_2) \rightarrow (17)$$

Adding eqn (13) & (17)

$$2E_0 = E_2 + E_2 (v_1/v_2)$$

$$= E_2 (1 + v_1/v_2)$$

$$E_0 = \frac{E_2}{2} (1 + v_1/v_2) \rightarrow (18)$$

when medium -1,

vacuum $v_1 = c$, $v_2 = v$

$$E_0 = \left(\frac{E_2}{2}\right)\left(1 + \frac{c}{v}\right)$$

subtracting eqn (17) from eqn (13)

$$E_1 = \left(\frac{E_2}{2}\right)\left(1 - v_1/v_2\right) \rightarrow (19)$$

$$E_1 = \left(\frac{E_2}{2}\right)\left(1 - \frac{c}{v}\right)$$

(12) Write short notes on i) Momentum and Radiation pressure ii) Cell phone Reception

i) Momentum and Radiation Pressure

- Electromagnetic waves carry energy and momentum
- Maxwell proved ... wave energy U
- Momentum are related by $P = U/c \rightarrow (1)$

As the electromagnetic waves carry momentum, they exert pressure when they are reflected or absorbed at the surface of a body. This is known as radiation pressure.

From Newton's second law, change in momentum related to force

$$F = \frac{\Delta P}{\Delta t} \rightarrow (2)$$

Intensity $I = \frac{Power}{area} = \frac{energy/time}{area}$

$$\Delta U = I \cdot A \cdot \Delta t \rightarrow (3)$$

eqn(1), momentum

$$\Delta P = \frac{\Delta U}{c} = \frac{I \cdot A \cdot \Delta t}{c} \rightarrow (4)$$

$$F = \frac{\Delta P}{\Delta t} = I \cdot A/c \rightarrow (5)$$

This is the relation for the total absorption of EM radiation.

ΔP - change in momentum

$$F = 2IA/c \rightarrow (6)$$

If the radiation is partly absorbed or completely reflected by the object, the magnitude of the force on area A varies between the values IA/c and $2IA/c$

Radiation Pressure

The force per unit area on an object due to EM radiation is the radiation pressure P_r.

from eqn (5) & (6)

$$P_r = F/A, \quad P_r = I/c$$

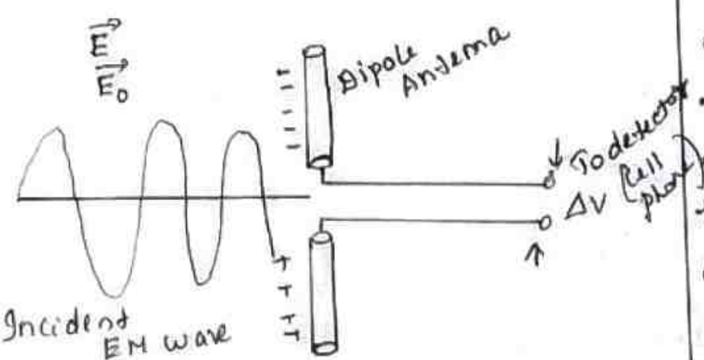
Total absorption of radiation

$$P_r = 2I/c$$

for total reflection back along the path.

---

ii) Cell phone Reception



- Contains a tiny low-power radio transmitter or antenna
- EM signal intensity decreases as the inverse square of the distance from the phone.
- Antenna's length is comparable to $\lambda/2$, λ - wavelength
- λ is short - cell phone antenna is very short
- EM signal induces a voltage across the wires of the antenna.
- Induced voltage is amplified.
- Low power signals emitted by the cell phone will be received and transmitted by the cell phone towers.
- Towers are another type of antenna.
- Cell phone transmits on one frequency and receive with other frequency.

13. Write short notes on i) Localized sources for Electromagnetic waves ii) Polarization iii) Producing electromagnetic waves

i) Localized Sources for EM waves

Electromagnetic waves can be produced either

- by accelerated electric charges
- by time varying electric currents

Magnetic field vector is mutually perpendicular to both electric field and the direction of wave propagation

$$E_y = E_0 \cos(\omega t - kx)$$

$$B_z = B_0 \cos(\omega t - kx)$$

In free space or vacuum the ratio between E_0 and B_0 is equal to the speed of electromagnetic wave is equal to speed of light c

$$c = E_0/B_0$$

$$V = E_0/B_0 < c$$

The energy of electromagnetic waves comes from the energy of the oscillating charge.

ii) Polarisation

The phenomenon by which the vibrations of the electric field vector of an electromagnetic wave to a particular plane is called polarization

The plane in which the electric field oscillates is defined as the plane of polarization.

Plane Polarized wave

If the variation of electric field is observed along the direction of propagation, tip of electric field vector appears to trace a straight line along vertical direction. In this vibration, E vector is confined to a single plane perpendicular to direction of propagation. Such wave is known as plane polarized wave.

Types of polarization

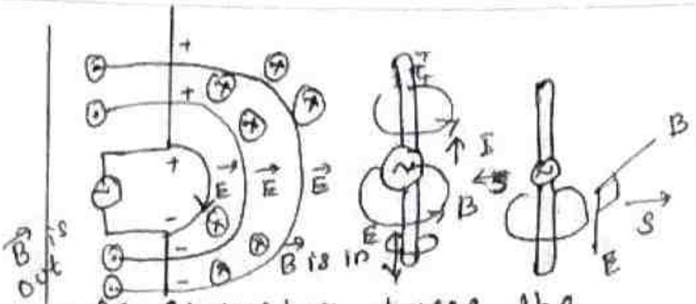
i) Linear Polarization

ii) Circular Polarization

iii) Elliptical Polarization

iii) Producing Electromagnetic waves

- Steady currents can produce electromagnetic waves.
- EM waves are the combination of electric and magnetic field produced by moving charges.
- Consider two conducting rods connected a source of alternating voltage.



- AC generator forces the charges to accelerate between two rods.
- Antenna compared to an oscillating electric dipole.
- Produced by the charge distribution on the wire.
- E and the charge distribution vary as the current changes.

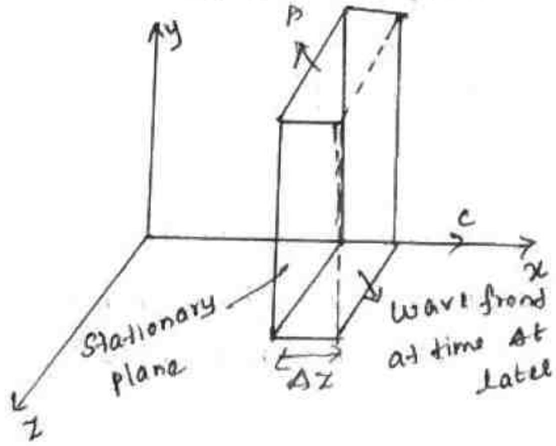
- The changing field propagates outward at the speed of light.
- The electric and magnetic fields are 90° out of the phase at all times.
- The electric and magnetic fields are closely related and propagate as an electromagnetic wave.
- Time varying electric field
- The result is the outward flow of electromagnetic wave energy at all times.

(14) Derive an expression for Electromagnetic Energy flow and Poynting vector

ii) Intensity of an EM wave in vacuum

i) Electromagnetic Energy flow and Poynting Vector

- EM waves transports energy from one region to another.



At a time Δt after this wave front moves a distance to the right side of the plane

$\Delta x = c\,\Delta t \quad \left(\therefore c = \frac{\Delta x}{\Delta t} \right)$

$\Delta V = A \cdot \Delta x = A \cdot c \cdot \Delta t$

If ΔU is the available energy

$\Delta U = u\,\Delta V = \left(\varepsilon_0 E^2\right)\left(A c \Delta t\right) \longrightarrow (1)$

u - energy density is equal to $\varepsilon_0 E^2$

$S = \frac{\Delta U}{A \cdot \Delta t} = \varepsilon_0 E^2 c \longrightarrow (2)$

$E = cB \quad , \quad c = 1/\sqrt{\mu_0 \varepsilon_0}$

Eqn (2) becomes

$S = \varepsilon_0 c^2 B^2 c \longrightarrow (3)$

$$S = \frac{\varepsilon_0 C B^2}{\varepsilon_0 \mu_0} \longrightarrow (4)$$

$(\because C\vec{B} = \vec{E})$ $\left(\because C^2 = \frac{1}{\varepsilon_0\mu_0}\right)$

or

$$S = \frac{CB\cdot B}{\mu_0} = \frac{EB}{\mu_0} \text{ (in vacuum)} \longrightarrow (5)$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \longrightarrow (6)$$

$$\vec{S} = \vec{E}\times\frac{\vec{B}}{\mu_0} \quad \left(\because \vec{B} = \mu_0\vec{H}\right)$$

$$\vec{S} = \vec{E}\times\vec{H}$$

Eqn (6) is the poynting vector in vacuum.

ii) Intensity of an EM wave in vacuum

Let us consider electric and magnetic field solutions

$$\vec{E}(x,t) = E_y \cos(\omega t - kx) \longrightarrow (8)$$

$$\vec{B}(x,t) = B_z \cos(\omega t - kx)$$

From eqn (6)

$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ becomes

$$\vec{S}(x,t) = \frac{1}{\mu_0} E_y \cos(\omega t - kx) \times B_z \cos(\omega t - kz) \longrightarrow (9)$$

$$S_x(x,t) = \frac{E_y B_z}{\mu_0}\cos^2(\omega t - kx) \longrightarrow (10)$$

$$= \frac{E_y B_z}{\mu_0}\left(\frac{1+\cos 2(\omega t - kx)}{2}\right) \longrightarrow (11)$$

The time average value of $\cos 2(\omega t - kx)$ is zero. So the average value of the poynting vector is

$$S_{average} = \vec{S}_x(x,t)$$

$$= \frac{E_x B_y}{2\mu_0} \longrightarrow (12)$$

or simply

$$S_{av} = \frac{E_y B_z}{2\mu_0} = \frac{E_y \cdot E_y}{2\mu_0 C} \longrightarrow (13)$$

$$= \frac{E_y \cdot E_y}{2\mu_0 \times \frac{1}{\sqrt{\mu_0\varepsilon_0}}}$$

$$S_{av} = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,E_y^2$$

$$S_{av} = \frac{1}{2}\sqrt{\frac{\varepsilon_0\varepsilon_0}{\mu_0\varepsilon_0}}\,E_y^2$$

$$S_{av} = \frac{\varepsilon_0}{\sqrt{\mu_0\varepsilon_0}}\,E_y^2$$

$$\text{Intensity } I = S_{av}$$

$$= \frac{1}{2}\varepsilon_0 C E_y^2 \longrightarrow (14)$$

This is the intensity of an EM wave in a vacuum

Intensity as localised sources as

$$I = \frac{\text{Power}}{\text{Area}} = \frac{P}{4\pi r^2}$$

unit : III

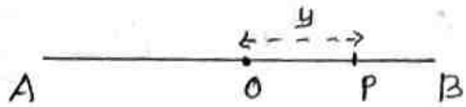
oscillations

Simple harmonic motion – resonance – analogy b/w electrical and mechanical oscillating systems – waves on a string – standing waves – travelling waves – Energy transfer of a wave – Sound waves – Doppler effect.

① Explain simple harmonic motion and discuss its characteristics

Definition : when the acceleration of particle is directly proportional to its displacement from its equilibrium position and it is always directed towards equilibrium position, then the motion of the particle is said to be simple harmonic motion.

Differential Eqn of S.H.M



- A particle executing S.H.M is called a harmonic oscillator.
- y – displacement
- t – time
- F – restoring force

$$F \propto -y$$

$$F = -ky \rightarrow (1)$$

k – Constant of proportionality called Spring factor.

- $a = \frac{d^2y}{dx^2}$, at instant t

Apply Newtons Second law

$$F = m\frac{d^2y}{dt^2} \rightarrow (2)$$

From eqns (1) & (2)

$$m\frac{d^2y}{dt^2} = -ky$$

$$m\frac{d^2y}{dt^2} + ky = 0$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

$$\boxed{\frac{d^2y}{dt^2} + \omega^2 y = 0} \rightarrow (3)$$

where, $\omega^2 = k/m$, ω – angular frequency.

Eqn (3) called as differential eqn of SHM.

- General solution of differential eqn for SHM

$$\boxed{y = A\sin(\omega t + \varphi)} \rightarrow (4)$$

A – amplitude, φ – initial phase

Angular harmonic motion

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \rightarrow (5)$$

$(\therefore \sin^2(\omega t+\varphi) + \cos^2(\omega t+\varphi) = 1$
or
$\cos^2(\omega t+\varphi) = 1 - \sin^2(\omega t+\varphi))$

Velocity and Acceleration

$$y = A\sin(\omega t + \varphi) \rightarrow (6)$$

Differentiating with respect to time t,

$$\frac{dy}{dt} = v = A\omega\cos(\omega t + \varphi)$$

$$\cos^2(\omega t+\varphi) = 1 - \sin^2(\omega t+\varphi)$$

$$v = A\omega\sqrt{1-\sin^2 \omega t+\varphi}$$

$$v = \omega\sqrt{A^2 - A^2\sin^2(\omega t+\varphi)}$$

$$v = \omega\sqrt{A^2 - y^2} \rightarrow (7)$$

$v_{max} = \omega A$ (at mean position)

$y = \pm A$ (extreme positions)

Acceleration

Differentiating the eqn (6) with respect to time t,

$$a = \frac{dv}{dt} = -A\omega^2 \sin(\omega t + \varphi)$$

$$\boxed{a = -\omega^2 y} \longrightarrow (8)$$

$a_{max} = \omega^2 A$

$a_{min} = 0$

Period of SHM

$\omega = 2\pi/T$, $T = \frac{2\pi}{\omega} \rightarrow (9)$

From eqn (8)

$\omega^2 = \frac{a}{y}$, $\omega = \sqrt{a/y}$, $\frac{1}{\omega} = \sqrt{y/a}$

Substituting in eqn (9)

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{y/a} \longrightarrow (10)$$

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

$T = 1/n \rightarrow (11)$

From eqn (10) and (11)

$\frac{2\pi}{\omega} = 1/n$, $\boxed{\omega = 2\pi n}$ $\rightarrow (12)$

Characteristic of Simple Harmonic motion

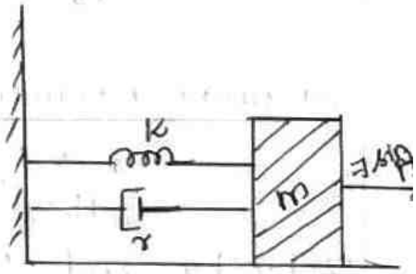
- The motion must be periodic.
- The motion is oscillatory
- SHM acted upon by restoring force
- If there is no air resistance or friction, the motion once started will continue indefinitely.

---

(2) Explain Forced vibration and discuss its characteristics

Definition: when a body A is maintained in the state of vibration by a periodic force of frequency ν other than its natural frequency (ν_0) of the body, the vibrations are called forced vibrations.

Differential Eqn for forced oscillations



. m - mass

. Forces acting on a particle

i) a restoring Force: Displacement acting in the opposite direction. It is given by $-ky$ where k is known as the restoring force constant.

ii) <u>a frictional force</u> : proportional to velocity but acting on the opposite direction. $-r\frac{dy}{dt}$

r - frictional force constant

iii) <u>the external periodic force</u>

$F\sin pt$ where F is the maximum value of the force and p is its angular frequency.

$$F' = -ky - r\frac{dy}{dt} + F\sin pt \longrightarrow (1)$$

Applying Newtons second law

$$F' = ma, \quad F' = m\frac{d^2y}{dt^2} \longrightarrow (2)$$

From eqns (1) & (2) $\left[a = \frac{d^2y}{dt^2}\right]$

$$m\frac{d^2y}{dt^2} = -ky - r\frac{dy}{dt} + F\sin pt$$

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + ky = F\sin pt \longrightarrow (3)$$

$$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{k}{m}y = F\sin pt$$

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = f\sin pt \longrightarrow (4)$$

where, $r/m = 2b$, $\frac{k}{m} = \omega^2$, $F/m = f$

Eqn (4) is called differential eqn of motion of the forced oscillation

The solution of differential eq (4)

$$y = A\sin(pt - \theta) \longrightarrow (5)$$

$$A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2p^2}} \longrightarrow (6)$$

$$\tan\theta = \frac{2bp}{\omega^2 - p^2}$$

$$\theta = \tan^{-1}\left[\frac{2bp}{\omega^2 - p^2}\right] \longrightarrow (7)$$

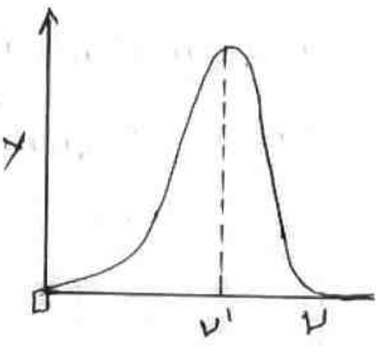
---

(3) <u>Discuss the Phenomenon of sharpness of resonance</u>

<u>Definition</u> :

The phenomenon of making a body vibrate with its natural frequency under the influence of another vibrating body with the same frequency is called resonance.

<u>Theory of resonant vibrations</u> ;



a) <u>condition of amplitude resonance</u>

$$A = \frac{f}{\sqrt{(\omega - p^2)^2 + 4b^2p^2}} \longrightarrow (1)$$

The amplitude is maximum $\sqrt{(\omega - p^2)^2 + 4b^2p^2}$ minimum

$$\frac{d}{dp}\left[(\omega - p^2)^2 + 4b^2p^2\right] = 0$$

$$2(\omega - p^2)^2(-2p)^2 + 4b^2(2p) = 0$$

$$\omega^2 - p^2 = 2b^2, \quad p = \sqrt{\omega^2 - 2b^2}$$

- When amplitude is maximum frequency $\frac{p}{2\pi}$ becomes $\frac{\sqrt{\omega^2 - 2b^2}}{2\pi}$ known as resonant frequency.

In the presence of damping $\frac{\sqrt{\omega^2 - b^2}}{2\pi}$, Absence of damping $\omega/2\pi$

$$p = \omega$$

Condition of Amplitude Resonance

using eqn (1), A is maximum

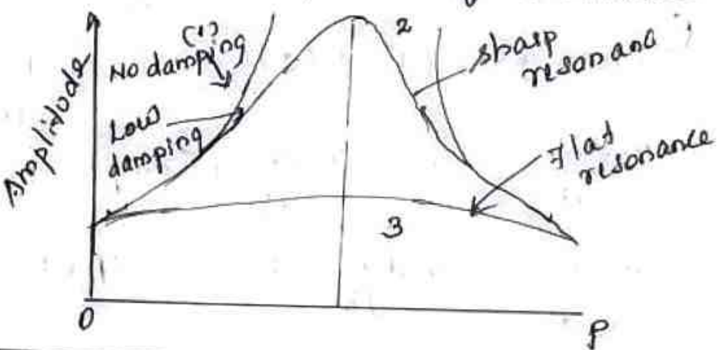
For negligible damping $b = 0$

$$A_{max} = \frac{f}{2bp_r}$$

$b \to 0$, $A_{max} \to \infty$

Sharpness of Resonance

The rate of change of amplitude with the change of forcing frequency on each side of resonant frequency is known as sharpness of resonance



(4) Discuss analogy between electrical and mechanical oscillating system

- In mechanical vibration, Particles both K.E and P.E.
- Total Energy is sum of these two energies

The eqn of motion $\frac{d^2y}{dt^2} + \omega^2 y = 0$

- Oscillator is free from damping

In case of spring

$$\boxed{f = \frac{1}{2\pi}\sqrt{K/m}}$$

$$\omega = 2\pi f = \frac{2\pi}{T} = \sqrt{K/m}$$

- The electrical oscillator system is described by charge q.

$$I = dq/dt$$

$$f = \frac{1}{2\pi}\sqrt{1/LC}$$

$$L\omega = \frac{1}{C\omega}$$

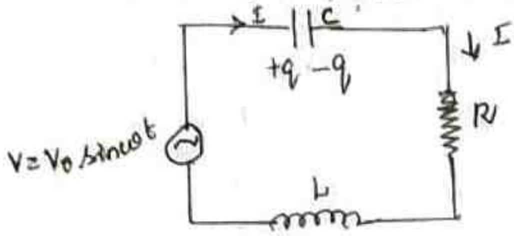
$$\omega^2 = \frac{1}{LC}$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Similarity between mechanical system and electrical system.

· It consists of capacitance (c) Inductance (L) Resistance (R)



$V = V_0 \sin \omega t$

$V_C = \frac{q}{c}$, $V_R = IR$ $V_L = L\frac{dI}{dt}$

$V_L + V_R + V_C = V$

$L\frac{dI}{dt} + IR + \frac{q}{c} = V_0 \sin \omega t$

since $I = dq/dt$

$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + q/c = V_0 \sin \omega t$

$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{CL} = \frac{V_0}{L}\sin \omega t \rightarrow (1)$

$I = dq/dt$

$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \sin \omega t$

$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{CL} = \frac{V_0}{L}\sin \omega t \rightarrow (1)$

The eqn(1) is similar to eqn of motion for a forced vibration.

$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = f \sin \omega t$

$2k = r/m$, $\omega_0^2 = s/m$, $f = F/m$

mass m is analogous to self inductance L, r to the electrical resistance R, compliance 1/s to the electrical capacitance C, force F to the voltage V_0, displacement x to the charge q, and velocity $\frac{dx}{dt}$ to the electrical current $\left(I = dq/dt\right)$.

(5) Deduce the wave eqn for Progressive wave

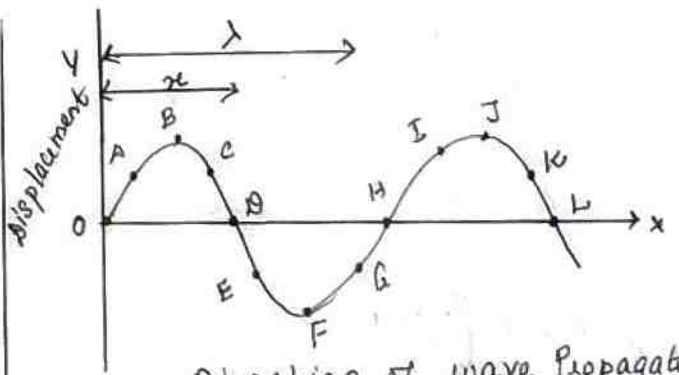
· on propagation of wave in a medium, the particles of medium execute simple harmonic motion.

· If a plane Progressive wave is Propagating in a medium along positive x-axis.

· Then the position of the particles O, A, B, C D ... are shown.

· The curve joining the positions represents the progressive wave.

· Let the particle begin to vibrate from O at time t = 0.



Direction of wave Propagation

y - displacement

t - time

SHM about O is

$$\boxed{y = A \sin \omega t} \rightarrow (1)$$

A - Amplitude

ω - angular velocity

(5)

If n is frequency of wave,

$$\omega = 2\pi n$$

The displacement of particle c at any time t will be same which was of particle o at time $\left(t - \frac{x}{v}\right)$.

The displacement of particle o at time $\left(t - \frac{x}{v}\right)$ can be obtained by substituting $\left(t - \frac{x}{v}\right)$ in place of t in eqn (1)

$$y = A \sin \omega \left(t - \frac{x}{v}\right) \longrightarrow (2)$$

T - Time period, λ - wavelength

$$\omega = \frac{2\pi}{T}$$

$$y = A \sin \frac{2\pi}{T}\left(t - x/v\right)$$

$$= A \sin 2\pi \left(\frac{t}{T} - \frac{x}{vT}\right)$$

$$vT = \lambda$$

$$y = A \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) \longrightarrow (3)$$

$\times$'y and $\div$ by λ

$$y = A \sin \frac{2\pi}{\lambda}\left(\frac{t\lambda}{T} - x\right)$$

$$y = A \sin \frac{2\pi}{\lambda}\left(vt - x\right)$$

$$y = A \sin \left(\frac{2\pi v t}{\lambda} - \frac{2\pi x}{\lambda}\right) \quad \left[\frac{\lambda}{T} = v\right]$$

$$y = A \sin \left(\frac{2\pi n v t}{v} - \frac{2\pi n x}{v}\right) \quad \left[\begin{array}{c} n\lambda = v \\ \frac{n}{v} = \frac{1}{\lambda} \end{array}\right]$$

The eqn(2) is also expressed as

$$y = A \sin \left(\omega t - \frac{\omega}{v} x\right)$$

$$[\because 2\pi n = \omega]$$

$\frac{\omega}{v} = \frac{2\pi}{\lambda} = k$ - Propagation constant

$$y = A \sin \left(\omega t - \frac{\omega}{v} x\right) \quad [\because 2\pi n = \omega]$$

$$\boxed{y = A \sin (\omega t - kx)} \longrightarrow (5)$$

Eqn(5) wave propagating along x-axis positive direction

$$\boxed{y = A \sin (\omega t + kx)} \longrightarrow (6)$$

Eqn(6) wave propagating along -ve direction in x-axis.

If φ is the phase difference between wave travelling along positive x-axis.

$$\boxed{y = A \sin \left[(\omega t - kx) + \varphi\right]} \longrightarrow (7)$$

Differential eqn of wave motion

$$y = A \sin \frac{2\pi}{\lambda}\left(vt - x\right)$$

$$\frac{dy}{dt} = \frac{2\pi v A}{\lambda} \cos \frac{2\pi}{\lambda}\left(vt - x\right) \longrightarrow (8)$$

$$\frac{dy}{dx} = -\frac{2\pi A}{\lambda} \cos \frac{2\pi}{\lambda}\left(vt - x\right) \longrightarrow (9)$$

Particle velocity

$$\frac{dy}{dt} = -v\frac{dy}{dx} \longrightarrow (10)$$

From the eqn (9)

$$\frac{d^2 y}{dx^2} = -A\left(\frac{2\pi}{\lambda}\right)^2 \sin \frac{2\pi}{\lambda}\left(vt - x\right) \longrightarrow (11)$$

From the eqn (8)

$$\frac{d^2 y}{dt^2} = -A\left[\frac{2\pi}{\lambda}\right]^2 v^2 \sin \frac{2\pi}{\lambda}\left(vt - x\right)$$

This is particle acceleration $\longrightarrow (12)$

Comparing (11) and (12)

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2} \longrightarrow (13)$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2}}$$

This is the differential eqn of wave motion.

It can be shown in case of Progressive waves, if, t is increased by δt and x by $v\delta t$

$$y' = a \sin \frac{2\pi}{\lambda}\left[v(t+\delta t) - (x + v\delta t)\right]$$

$$y' = a \sin \frac{2\pi}{\lambda}\left[vt + v\delta t - x - v\delta t\right]$$

$$y' = a \sin \frac{2\pi}{\lambda}(vt - x) = y \longrightarrow (14)$$

Thus in a time δt, the wave advances through $v\delta t$.

v - velocity of the wave

Charecteristics of Progressive wave

i) Each particle of the medium executes vibration about its mean position.

ii) The particles of the mediom vibrate with the same amplitude about their mean positions.

iii) The phase of every particle changes from 0 to 2π.

iv) No particle remains permanently at rest.

v) Transverse progressive waves are characterised by crests and toughs.

vi) Longitudinal waves are characterised by compressions and rarefactions.

vii) transfer of energy across the medium in the direction of propagation of progressive wave

6) Derive the wave eqn for stationary waves or standing waves

Definition :

when two progressive waves of same amplitude and wavelength travelling along a straight line in opposite directions Superimpose on each other, stationary waves are formed.

Analytical method

consider a Progressive wave of amplitude a and wavelength λ in x axis

$$y_1 = a \sin 2\pi\left(\frac{t}{T} - x/\lambda\right)$$

$$y_2 = a \sin 2\pi\left[\frac{t}{T} + \frac{x}{\lambda}\right]$$

According to principle of Super position, the resultant displacement is

$$y = y_1 + y_2$$

$$= a\left[\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + \sin 2\pi\left(\frac{t}{T} - x/\lambda\right)\right]$$

$$= \left[\sin\left(\frac{2\pi t}{T}\right) - \frac{2\pi x}{\lambda} + \sin\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)\right]$$

$$= a\left[2\sin\frac{2\pi t}{T}\cos\frac{2\pi x}{\lambda}\right]$$

$$\boxed{\sin(A-B)+\sin(A+B) = 2\sin A\cos B}$$

$\therefore y = 2a\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi t}{T}$

This is the eqn of a stationary wave.

i) At points $x = 0, \lambda/2, \lambda, 3\lambda/2$ the values of $\cos\frac{2\pi x}{\lambda} = \pm 1$

$A = \pm 2a$. These points of the resultant amplitude is maximum. They are called antinodes.

ii) At points $x = \lambda/4, \frac{3\lambda}{4}, 5\lambda/4$

$\cos\frac{2\pi x}{\lambda} = 0$

$A = 0$. The resultant amplitude is zero called nodes.

iii) $t = 0, T/2, T, \frac{3T}{2}, 2T \ldots$

$\sin\frac{2\pi t}{T} = 0$, displacement is zero.

iv) $t = T/4, \frac{3T}{4}, \frac{5T}{4}$ etc..

$\sin\frac{2\pi t}{T} = \pm$, displacement is maximum.

<u>Characteristics of stationary waves</u>

i) The waveform remains stationary.

ii) Nodes and antinodes are formed alternately.

iii) The points where displacement is zero are called nodes and the points where the displacement is maximum are called antinodes.

iv) Pressure changes are maximum at nodes and minimum at antinodes

v) All the particles except those at the nodes and minimum at antinodes.

vi) Amplitude of each particle is not the same.

vii) The velocity of the particles at the nodes is zero.

viii) There is no transfer of energy.

(7) Discuss about Energy Transfer of a wave

· The mechanical energy is transferred through the vibration of the string.

T - tension, m - mass

l - length

· length dx is considered then its mass $m.dx$

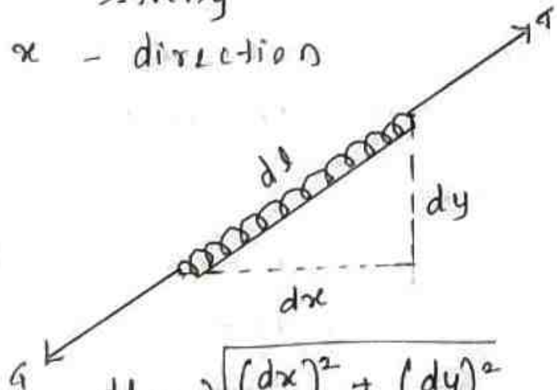
As the string is vibrating, the K.E of this small element

$$dK = \frac{1}{2} m (\text{velocity})^2$$

$$dK = \frac{1}{2} m \cdot dx \left(\frac{dy}{dt}\right)^2 \longrightarrow (1)$$

dy - vertical displacement of a string

x - direction



$$dl = \sqrt{(dx)^2 + (dy)^2}$$

$$= dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$dl = dx\left(1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right) \longrightarrow (2)$$

String has expanded by an amount

$$\Delta l = dl - dx = \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

If the string is vibrating with displacement

$$y(x,t) = A\cos(\omega t - kx) \longrightarrow (4)$$

(9)

The potential energy is

$$dU = T \cdot \Delta l$$

$$= \frac{1}{2} T \left(\frac{dy}{dx}\right)^2 dx \longrightarrow (5)$$

Substituting (4) in (5)

$$dU = \frac{1}{2} T k^2 A^2 \sin^2(\omega t - kx) \cdot dx$$

$$dU = \frac{1}{2} m A^2 \omega^2 \sin^2(\omega t - kx) dx \longrightarrow (6)$$

where

$$v = \frac{\omega}{k}, \quad v = \sqrt{T/m}, \quad T = mv^2$$

$$Tk^2 = T \cdot \frac{\omega^2}{v^2} = T \frac{\omega^2}{(T/m)} = m\omega^2$$

If we substitute eqn (4) in (1)

$$dK = \frac{1}{2} m A^2 \omega^2 \sin^2(\omega t - kx) \cdot dx \longrightarrow (7)$$

Comparing eqn (4) and eqn (6)

$$dU = dK \longrightarrow (8)$$

The total energy is

$$dE = dU + dK = 2dK$$

$$= \frac{1}{2} m A^2 \omega^2 \sin^2(\omega t - kx) \cdot dx$$

(or)

$$dE = m(A\omega)^2 \sin^2(\omega t - kx)\, dx \longrightarrow (10)$$

The quantity $\frac{dE}{dx}$ is called the linear density.

$$\left(\frac{dE}{dx}\right)_{\text{average}} = d\vec{E} = \frac{1}{2} m (A\omega)^2 \longrightarrow (11)$$

This relation is called as average energy density $d\vec{E}$

As the average power transmitted by the wave

$$\vec{P} = \left(\frac{dE}{dt}\right)_{Average}$$

Eqn (11) becomes

$$(dE)_{Average} = \frac{1}{2} m A^2 \omega^2 dx \longrightarrow (12)$$

$$\vec{P} = \frac{1}{2} m A^2 \omega^2 dx/dt$$

$$\vec{P} = 1/2\, m A^2 \omega^2 . v \longrightarrow (13)$$

where, $v = \frac{dx}{dt}$ is the wave velocity.

(8) Explain about Doppler effect and its Applications.

Doppler Effect

The phenomenon of the apparent change in the frequency of the sound due to relative motion between the source of sound and the observer is called Doppler effect.

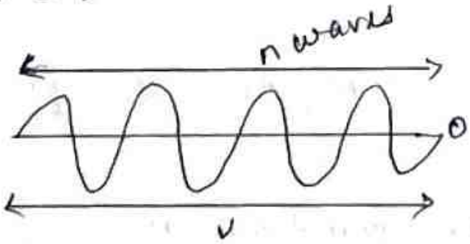
i) Both source and observer at rest

S & O - Source and observer

n - frequency of sound

v - velocity of sound

SO = v



original wavelength $\lambda = \frac{v}{n}$ $\longrightarrow (1)$

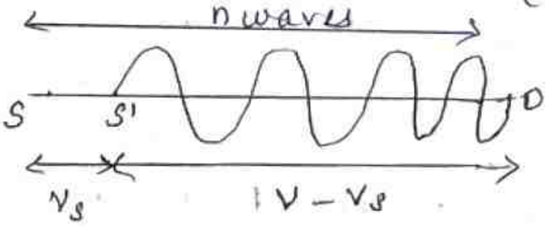
original frequency

$$n = v/\lambda \longrightarrow (2)$$

ii) when the Source moves towards the stationary observer

$SS' = v_s$

n waves emitted by the source will occupy a distance $(v - v_s)$



∴ apparent wavelength of Sound

$$\lambda' = \frac{v - v_s}{n} \longrightarrow (3)$$

The apparent frequency

$$n' = \frac{v}{\lambda'} = \left(\frac{v}{v - v_s}\right) n \longrightarrow (4)$$

iii) when the sources moves away from the stationary observer

The apparent frequency

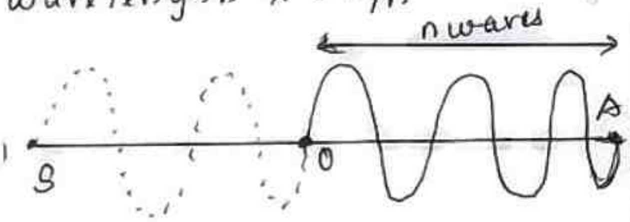
$$n' = \frac{v}{\lambda'} = \left(\frac{v}{v-(-v_s)}\right)n$$

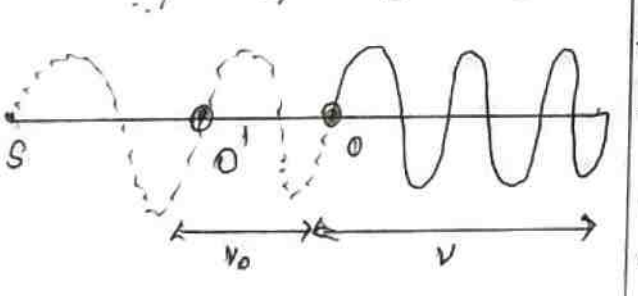
$$n' = \left(\frac{v}{v+v_s}\right)n \longrightarrow (5)$$

There is apparent change in wavelength and hence the frequency changes.

iv) Source is at rest observer in motion

Source S emits n waves per second having a wavelength $\lambda = v/n$





V) when the observer moves towards the stationary

stationary source with velocity v_o.

observer will reach the point O'

$OO' = v_o$

The apparent frequency of sound

$$n' = \frac{v+v_o}{\lambda} = n + \left(\frac{v_o}{v}\right)n$$

The apparent frequency of sound

$$n' = \left(\frac{v+v_o}{v}\right)n \longrightarrow (6)$$

vi) when the observer moves away from the stationary source

apparent frequency of sound

$$n' = \left(\frac{v+(-v_o)}{v}\right)n$$

apparent frequency of sound

$$n' = \left(\frac{v-v_o}{v}\right)n \longrightarrow (7)$$

Applications of Doppler effect

i) To measure the speed of an automobile

- An electromagnetic wave is emitted by a source attached to a police car. The wave is reflected by a moving vehicle which acts as a moving source.
- Shift in the frequency of reflected wave.
- Frequency shift using beats, the speeding vehicles are trapped by the police.

ii) RADAR (Radio detection and ranging)

• RADAR sends high frequency radiowaves towards an aeroplane.
• The reflected waves are detected by the receiver of the radar station.
• The difference in frequency is used to determine the speed of an aeroplane.

iii) SONAR (Sound navigation and ranging)

Sound waves generated from a ship fitted with SONAR are transmitted in water towards an submarine.

• Frequency of the reflected waves is measured.
• Speed of the submarine is calculated.

iv) Blood flow meter

Ultrasonic sounds are transmitted towards organs the frequency change in reflected waves used to measure blood flow rate.

v) Tracking a satellite

The frequency of radio waves emitted by a satellite decreases as the satellite passes away from the earth.
• The frequency received by the earth station combined with a constant frequency generated the beat frequency.
• Satellite is tracked.

vi) Stars moving towards the earth or away from the earth

• There is an apparent change in wavelength of spectral lines emitting by a moving star.
• Spectral shift enables the velocity of star to be the computed along the line of sight.

Unit - III

Oscillations, Optics and Lasers

Optics

Reflection and refraction of light waves - total internal reflection - interference Michelson interferometer - Theory of air wedge and experiment.

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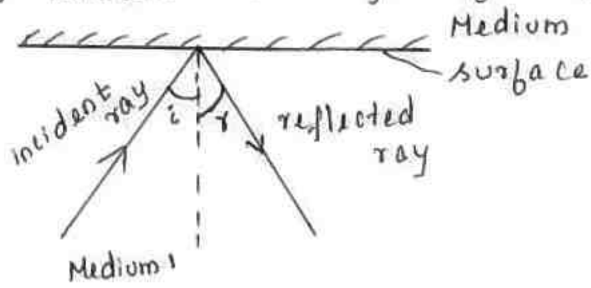
1) Write notes on i) Reflection of light waves ii) Refraction of light waves iii) Total internal

i) Reflection of light waves

i) The phenomenon where the incident light falling from one medium on a surface of another medium is sent back to the same medium is known as reflection.

ii) The angle between the incident ray and the normal to the surface is known as angle of incidence (i).

iii) The angle between the reflected ray and the normal of the surface is known as angle of reflection (r).



Laws of Reflection

i) Incident ray, normal and reflected ray lie in the same plane.

ii) The angle of incidence is equal to the angle of reflection $\angle i = \angle r$

ii) Refraction of Light waves

Refraction is the phenomenon in which light travels from one medium (air) to another medium (glass). The direction of light changes due to change in medium

Laws of refraction

i) The incident ray, the refracted ray and the normal at a point of seperation of two media lie in the same plane.

ii) For any two medium, the ratio of sine of angle of incidence to sine of angle of refraction is constant. It is known as Snell's law.

$$\frac{\sin i}{\sin r} = \text{Constant}$$

$$\boxed{\mu = \frac{\sin i}{\sin r}}$$

• constant is called as refractive index (μ)

Significance of refractive index

Refractive index μ = $\dfrac{\text{Velocity of light in vacuum (c)}}{\text{Velocity of light in medium (v)}}$

$$\mu = \frac{c}{v}$$

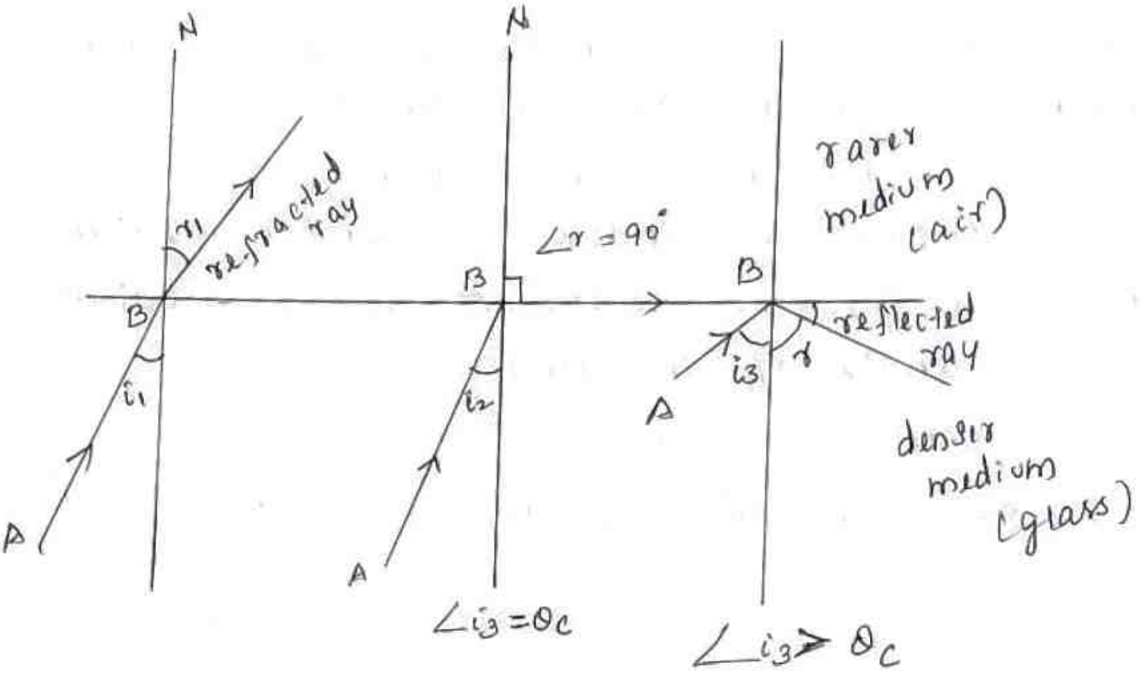
i) when light travels from rarer medium to denser medium, $\angle i$ is greater than $\angle r$

So refractive index always greater than 1.

ii) when light travels from denser medium to rarer medium, $\angle i$ is less than $\angle r$. So refractive index be less than 1.

ii) Refractive index for vacuum is unity (1).

iii) Total Internal Reflection



i) when light passes from denser medium to rarer medium, the refracted ray bends away from the normal.

ii) A ray AB incident at $\angle i$ and refracted at $\angle r_1$, as angle of incidence increases, angle of refraction also increases.

iii) For a particular value of angle of incidence $\angle i_2$, the refracted ray travels along the surface of seperation between the two medium. $\angle r$ becomes $90°$, then the angle of incidence is called as critical angle θ_c.

iv) For the angle of incidence greater than critical angle ($\angle i_3 > \theta_c$), the ray cannot pass into second medium but completely gets reflected in the same medium.

v) Thus a ray travelling from denser medium to a rarer medium is reflected into denser medium if angle of incidence is more than the critical angle of medium.

Critical angle - Definition

The angle of incident at which the refracted ray just graze surface between denser and rarer media is called critical angle.

$$n_i \sin i = n_2 \sin r$$

n_1 - refractive index of denser medium

n_2 - refractive index of rarer medium.

i - angle of incident

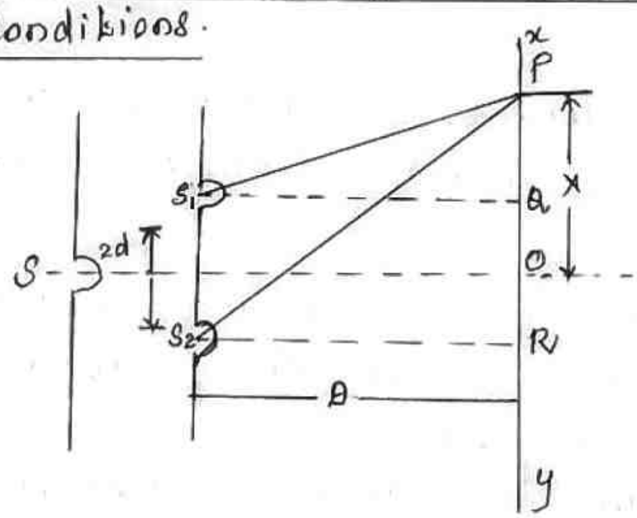
r - angle of refraction

$i = \theta_c$, $r = 90^\circ$, $n_1 \sin \theta_c = n_2 \sin 90^\circ$ $(\because \sin 90^\circ = 1)$

$$\sin \theta_c = \frac{n_2}{n_1}$$

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

(2) Describe about Theory of interference fringes and write the conditions.



. Consider monochromatic source S.

. S_1, S_2 - Pin holes, equidistant from S.

- 2d - distant
- Let a screen XY be placed at a distant D parallel to S_1, S_2.
- Point P at a distance x from O.

From right angled triangle S_1QP

$$(S_1P)^2 = (S_1Q)^2 + (QP)^2 \qquad [\because QP = (x-d)]$$

$$(S_1P)^2 = D^2 + (x-d)^2 \longrightarrow (1)$$

Right angle triangle S_2RP

$$(S_2P)^2 = (S_2R)^2 + (RP)^2 \qquad [RP = x+d]$$

$$(S_2P)^2 = D^2 + (x+d)^2 \longrightarrow (2)$$

$$(S_2P)^2 - (S_1P)^2 = D^2 + x^2 + d^2 + 2dx - D^2 - x^2 + 2dx - d^2$$

$$(S_2P)^2 - (S_1P)^2 = 4xd$$

$$(S_2P - S_1P)(S_2P + S_1P) = 4xd \longrightarrow (3)$$

In young's experiment, D is thousand times greater than 2d or x so that if $(S_2P + S_1P)$ is replaced by 2D,

$$(S_2P - S_1P)\,2D = 4xd$$

$$(S_2P - S_1P) = \frac{4xd}{2D} = \frac{2xd}{D} \longrightarrow (4)$$

Position and spacing of fringes

1) Bright fringes : $S_2P - S_1P = n\lambda$, $n = 0, 1, 2$

$$\frac{2xd}{D} = n\lambda$$

$$\boxed{x = \frac{n\lambda D}{2d}} \longrightarrow (5)$$

$$n=1, \quad x_1 = \frac{\lambda D}{2d}, \quad n_2, \; x_2 = \frac{2\lambda D}{2d}$$

$$n=3, \quad x_3 = \frac{3\lambda D}{2d}, \quad n_n, \; x_n = \frac{n\lambda D}{2d} \quad \longrightarrow (6)$$

The distance between any two consecutive bright fringes

$$x_2 - x_1 = \frac{2\lambda D}{2d} - \frac{\lambda D}{2d} = \frac{D\lambda}{2d} \longrightarrow (7)$$

2) Dark Fringes

$$(S_2P - S_1P) = (2n+1)\lambda/2 \quad , \quad n = 0,1,2,3$$

$$\frac{2xd}{D} = \frac{(2n+1)\lambda}{2}$$

$$x = \frac{(2n+1)\lambda D}{4d} \longrightarrow (8)$$

when $n=0, \; x_0 = \lambda D/4d$

$$n=1, \quad x_1 = \frac{3\lambda D}{4d}$$

$$n=2, \quad x_2 = \frac{5\lambda D}{4d}$$

$$n=n, \quad x_n = \frac{(2n+1)\lambda D}{4d}.$$

The distance between any two consecutive dark fringes

$$x_2 - x_1 = \frac{5\lambda D}{4d} - \frac{3\lambda D}{4d} = \frac{2\lambda D}{4d} = \frac{\lambda D}{2d} \longrightarrow (9)$$

Spacing between any two consecutive maximum or minima is the same. This expressed by β

$\left(\beta = \frac{\lambda D}{2d}\right)$ and it is known as fringe width.

Conditions for Interference of light.

i) Conditions for sustained interference

- Sources should be coherent.
- Sources should emit continuous waves of the same wavelength and time period.

ii) Conditions for observations.

- Seperation between two sources should be small.
- Distance 'D' between two sources and the screen should be large.
- The background should be dark.

iii) Conditions for good contrast

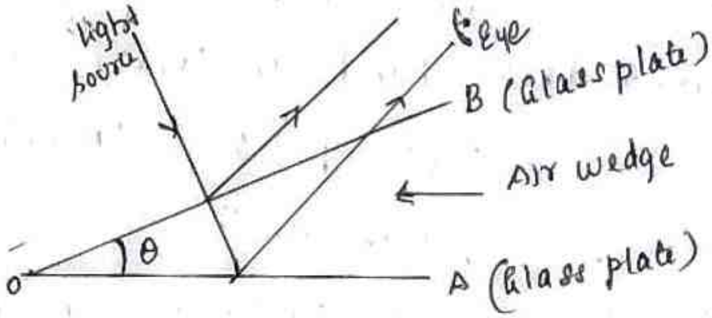
- Sources must be narrow.
- Sources should be monochromatic.
- Amplitudes should be equal or nearly equal.

③ Describe about Theory of Air wedge

Definition:

A wedge shaped (v-shaped) air film enclosed in between two glass plates is called air wedge.

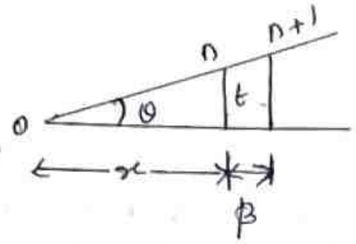
Theory



- A & B - glass plates
- Light from a monochromatic light source is made to fall perpendicularly on the film.
- The incident rays of light is partially reflected from the upper surface of the

air film and partially reflected from the lower surface of the air film.

• Two reflected rays will interfere and a large number of straight alternative bright and dark fringes are formed.



For air film, refractive index of the film $\mu = 1$

$\cos r = 1$, $r = 0$, $\cos \theta = 1$

$$2t = n\lambda \longrightarrow (2)$$

x - distance of the n^{th} dark band from the edge of contact O.

$$\frac{t}{x} = \tan\theta$$

$$\frac{t}{x} = \theta \quad (\because \theta \text{ is very small } \tan\theta \approx \theta)$$

$$t = x\theta \longrightarrow (3)$$

substituting eqn (3) in eqn (2); for the n^{th} dark band

$$2x\theta = n\lambda$$

Next dark band ie) $(n+1)^{th}$ dark band

$$2(x+\beta)\theta = (n+1)\lambda \longrightarrow (5)$$

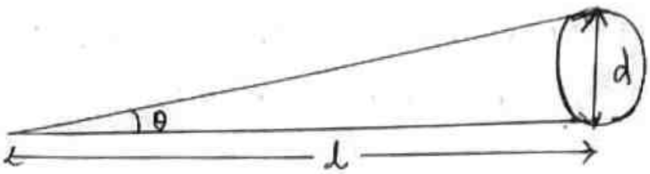
β - fringe width

Subtracting eqn (4) from eqn (5)

$$2\beta\theta = \lambda$$

$$\boxed{\beta = \lambda/2\theta} \longrightarrow (6)$$

Thickness of a thin wire and very thin foil



d - thickness

l - distance from the edge of contact

$\tan\theta = d/l \quad (\because \tan\theta \approx \theta)$

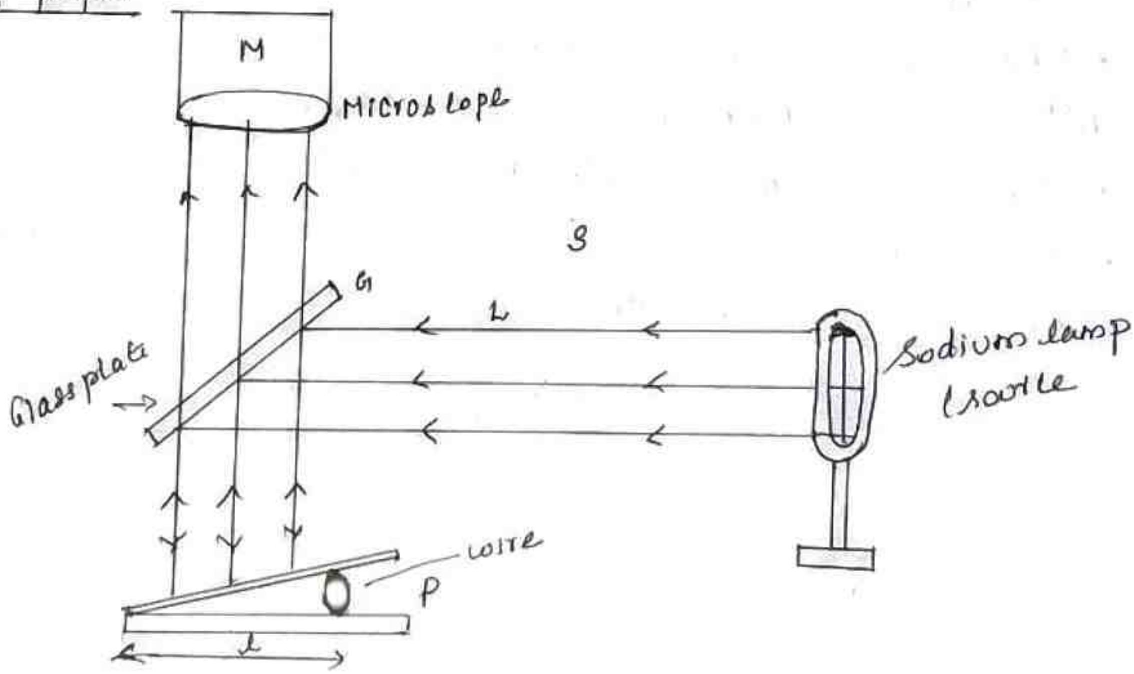
$\theta = d/l \longrightarrow (7)$

Substituting eqn (7) in (6)

$\beta = \frac{\lambda}{2d/l} = \frac{\lambda l}{2d}$

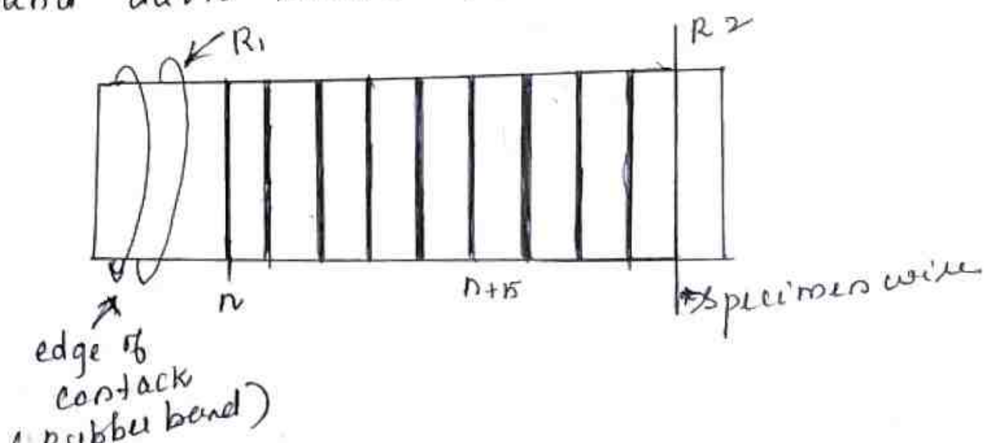
$$\boxed{d = \frac{\lambda l}{2\beta}}$$

(4) Describe air wedge experiment by using determination of diameter (thickness) of a wire or thickness of a thin sheet of paper.



(9)

- Air wedge is formed by keeping two optically plane glass plates in contact along one of the edges, and a thin wire near other end, parallel to the contact edges of the glass plates.
- Inclined at a very small angle θ.
- Glass plate kept inclined at angle 45° to the horizontal.
- Interference takes between top and bottom surface.
- bright and dark bands viewed.



- Reading on the horizontal scale of the microscope is noted.
- The cross wire is made to coincide with successive 5th fringes (n+5, n+10 ... n+40).
- Average fringe width β is determined.
- Using microscope, distance l between the edge of the contact and the wire is also measured.

$$\boxed{d = \frac{l\lambda}{2\beta} \text{ m}}$$

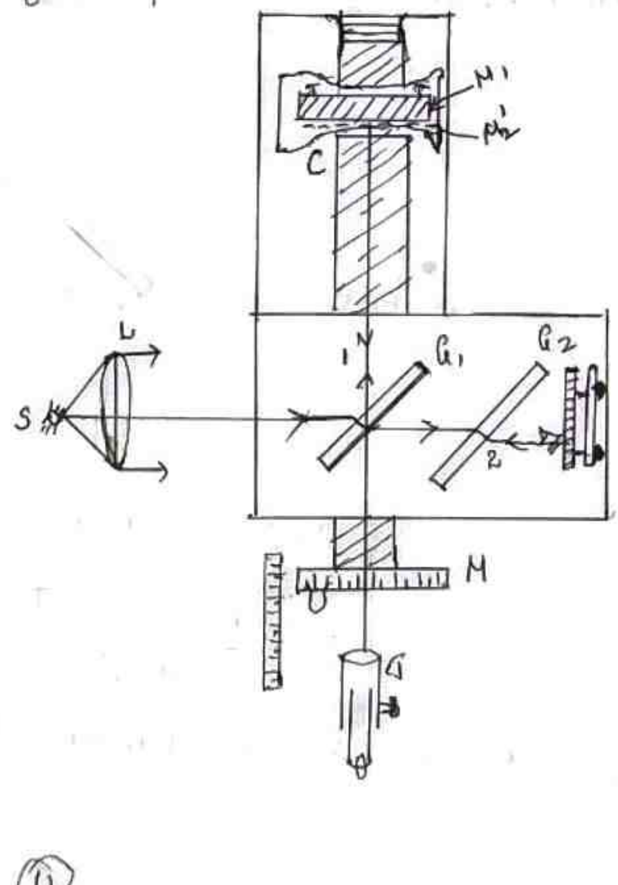
S. No	order of fringes	Microscope reading $\times 10^{-2}$ m	Width of 10 fringes m	Band width β m

Mean : ______

(5) Explain the construction, Determination of a thickness of a thin transparent sheet of Michelson's interferometer.

Principle : Two interfering beams are formed by splitting the light from a source into two parts by partial reflection and refraction. These beams are sent in two perpendicular directions and they are finally brought together after reflection from plane mirrors to produce interference fringes.

Construction :



- Two mirrors - M_1, M_2
- Micrometer screw - M
- Mirror M_2 is fixed.
- Thickness placed at an angle of 45° to the incident beam.
- plate G_1 is semi silvered on its back side and acts as a beam splitter.
- G_2 - Compensating glass plate.
- S - Monochromatic light source.

Working

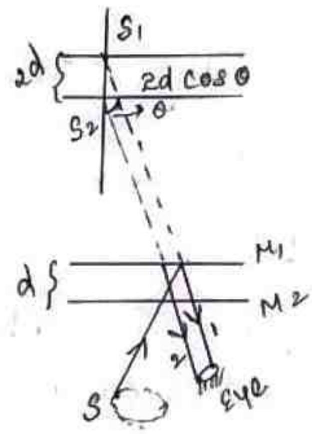
- Light from source S parallel by means of a collimating lens.
- Light beam divided into two parts.
- One part of the light reflected to travel towards mirror M_1.
- Other part of the light is transmitted towards M_2.
- Light rays falling on mirrors M_1 and M_2.
- mirror M_1 directly together with a virtual image of M_2, denoted by M_2'.
- Interference fringes be straight, circular or parabolic.
- Depending, Path difference, M_1 and M_2'.

Formation of fringes.

For maximum intensity in the fringes

$$2d \cos\theta + \lambda/2 = n\lambda$$

where $n = 0, 1, 2 \ldots$

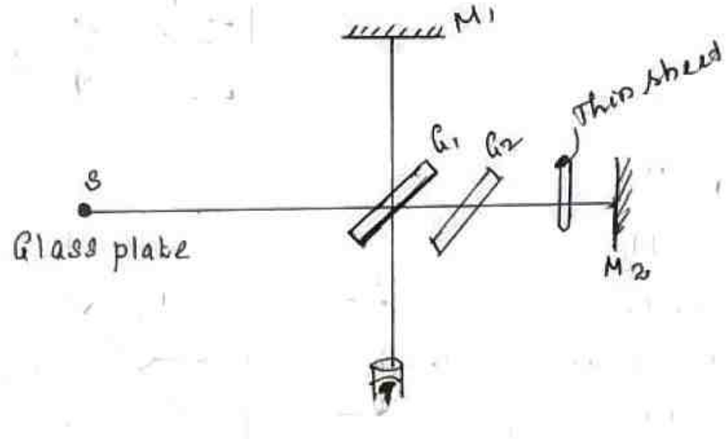


Wavelength Determination

$$M_1 = d = n\lambda/2$$

$$\boxed{\lambda = \frac{2d}{n}}$$

Determination of Thickness of a thin transparent sheet



$$2(\mu - 1)t = n\lambda$$

$$\boxed{t = \frac{n\lambda}{2(\mu - 1)}}$$

If μ, n and λ are known, thickness t can be calculated.

Unit - III

Oscillations, Optics and Lasers

Lasers

Theory of Laser - characteristic - spontaneous and stimulated emission - Einstein's coefficients - population inversion - Nd - YAG laser CO_2 laser - Semiconductor laser - Basic applications of lasers in industry.

(1) **Derive an expression for Einstein's A and B Coefficients (Derivation)**

Einstein introduced two coefficients known as Einstein's A and B coefficients to describe the absorption and emission process.

When an electromagnetic signal is incident on an atom, three different Process occur

i) Stimulated absorption

ii) Spontaneous emission

iii) Stimulated emission

Stimulated absorption

The atom in the lower energy state E_1 absorbs radiation and is excited to the higher energy level E_2. This process is called stimulated or induced absorption

$$N_{ab} \propto N_1 Q$$

Q - energy density

N_1 - number of atoms

The number of stimulated absorption transitions occuring per unit time is given by

$$N_{ab} = B_{12} N_1 Q \longrightarrow (1)$$

where,

B_{12} is a proportionality constant.

This process is an upward transition

By emitting a photon of energy $h\nu$ in two ways

a) Spontaneous emission

b) Stimulated emission

Spontaneous emission

The atoms in the excited state E_2 return to lower energy state E_1 by emitting a photon of energy energy $h\nu$ without the influence of any external agency. This emission of light radiation is known spontaneous emission.

The rate of spontaneous emission is directly proportional to the number of atoms in the excited energy state (N_2)

$$N_{sp} \propto N_2$$

$$N_{sp} = A_{21} N_2 \longrightarrow (2)$$

A_{21} - Proportionality constant.

This process is a downward transition.

Stimulated emission

If the light photon is incident on the atom in the excited energy state, the photon triggers the excited atom to make transition to lower energy E_1 along with emission of photons. This kind of emission of light radiation is stimulated emission.

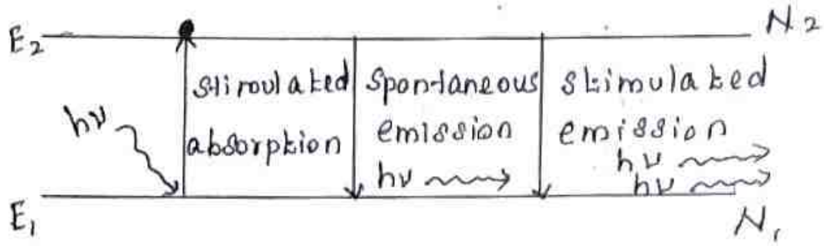
$$N_{st} \propto N_2 Q$$

The number of transitions per second

$$N_{st} = B_{21} N_2 Q \longrightarrow (3)$$

B_{21} - Proportionality constant

This process is also downward transition.



The proportionality constants A_{12}, B_{12}, and B_{21} are known as Einstein's coefficients A and B.

Under equilibrium condition, the number of downward and upward transitions per second are equal

$$N_{sp} + N_{st} = N_{ab} \longrightarrow (4)$$

Substituting eqns (1), (2), (3) in eqn (4)

$$A_{21} N_2 + B_{21} N_2 Q = B_{12} N_1 Q \longrightarrow (5)$$

Rearranging the eqn (5), we have

$$B_{12} N_1 Q - B_{21} N_2 Q = A_{21} N_2$$

$$Q\left(B_{12} N_1 - B_{21} N_2\right) = A_{21} N_2$$

$$Q = \frac{A_{21} N_2}{B_{12} N_1 - B_{21} N_2} \longrightarrow (6)$$

Dividing numerator and denominator by B_{21}, N_2

$$Q = \frac{\dfrac{A_{21} \cancel{N_2}}{B_{21} \cancel{N_2}}}{\dfrac{B_{12} N_1}{B_{21} N_2} - \dfrac{\overset{1}{\cancel{B_{21} N_2}}}{\cancel{B_{21} N_2}}}$$

$$Q = \frac{A_{21}}{B_{21}} \frac{1}{\left(\frac{B_{12}}{B_{21}}\right) \frac{N_1}{N_2} - 1} \longrightarrow (7)$$

Substituting $\frac{N_1}{N_2} = e^{h\nu/KT}$

$$Q = \frac{A_{21}}{B_{21}} \frac{1}{\left(\frac{B_{12}}{B_{21}}\right) e^{h\nu/KT} - 1} \longrightarrow (8)$$

Planck's radiation formula for energy distribution is given by

$$Q = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/KT} - 1} \longrightarrow (9)$$

Comparing the eqns (8) and (9)

$$\frac{B_{12}}{B_{21}} = 1$$

$$B_{12} = B_{21} \longrightarrow (10)$$

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3} \qquad \left(\nu = c/\lambda\right)$$

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h \cancel{c^3}}{\cancel{c^3}\lambda^3}$$

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h}{\lambda^3} \longrightarrow (11)$$

Since $B_{12} = B_{21}$, Einstein's coefficients are termed on as A and B coefficients.

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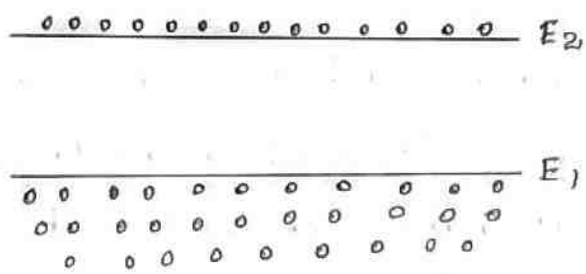
(2) write short notes on i) Population inversion ii) characteristics of Laser Light

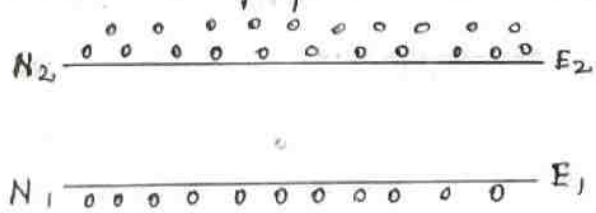
i) Population inversion

It is a situation in which the number of atoms in higher energy state is more than that in lower energy state.



Normal Condition

The state of achieving more number of atoms in higher energy state than the that of lower energy state is known as population inversion



After population inversion

(5)

Conditions for population inversion

- There must be atleast two energy levels
- There must be a source to supply the energy to the medium.
- The atoms must be continuously raised to the excited state.

Active Medium or Material

A medium in which population inversion above threshold inversion density is achieved is known as active medium. It is also called active material

The inversion density which is just enough to compensate for the losses in the medium is called threshold inversion density.

Pumping action

The process to achieve population inversion in the medium is called pumping action.

Methods for Pumping action

Four pumping action, They are

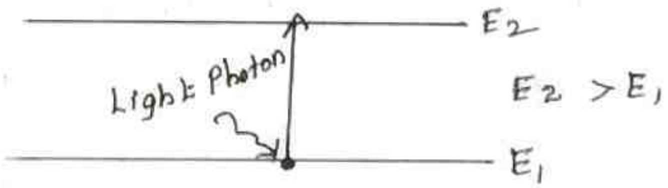
i) optical pumping (excitation by Photons)

ii) Electrical discharge (excitation by electrons)

iii) Direct Conversion

iv) Inelastic Collision between atoms

i) optical pumping

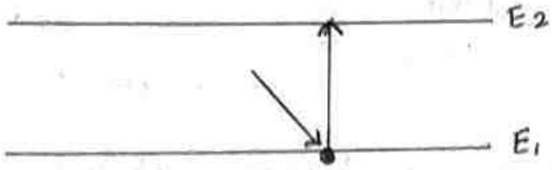


When the atoms are exposed to light radiation of energy $h\nu$, atoms in the lower energy state absorb these radiation and go to an excited state. This is known as optical pumping.

It is used in solid state lasers like ruby and ND-YAG lasers.

ii) Electrical discharge

During the collision, the energy of the electrons is transferred to gas atoms. Thereby atoms gain energy and go to excited state. This results in population inversion. This is known as electrical discharge.



Electrical discharge method

The energy transfer is represented by the equation

$$\boxed{A + e^* \longrightarrow A^* + e}$$

A - Gas atom (or molecule) in ground state

A^* - Same gas atom in excited energy state

e^* - Electron with more kinetic energy.

e - Same electron with less energy.

This method of pumping is used in gas lasers like argon and CO_2 lasers.

iii) Direct conversion

During the recombination process, the electrical energy is directly converted into light energy.

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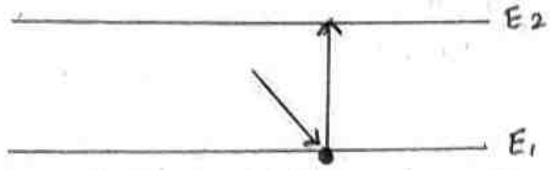

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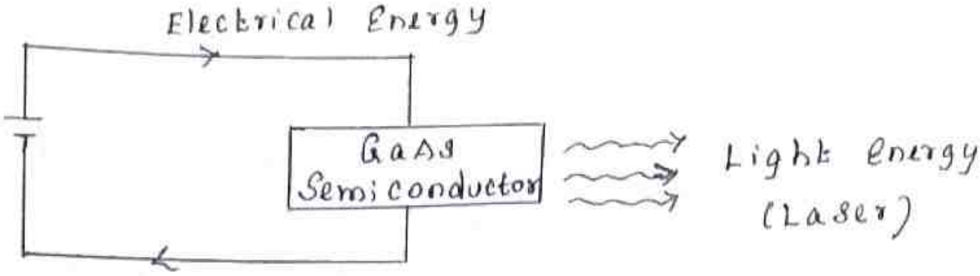
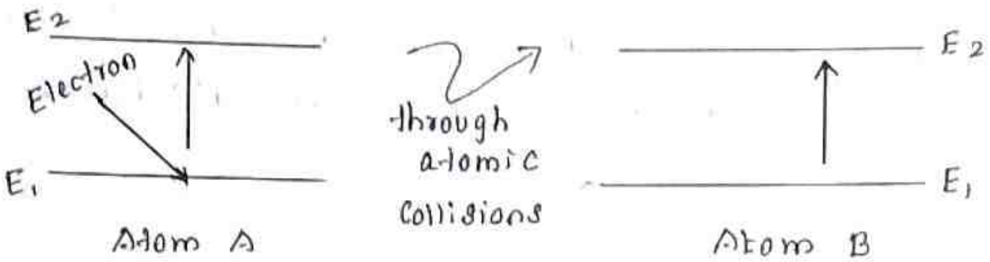


iv) Inelastic collision between atoms

- A Combination of two gases (A & B) is used.
- The excited energy levels of gases of A and B nearly Coincides each other.



During the electrical discharge, atoms of gas A are excited to higher energy states A^* due to collision with the electrons.

$$\boxed{A + e^* \longrightarrow A^* + e}$$

e^* - Electron with more kinetic Energy

e - same electron with less energy.

Due to inelastic collision, B atoms gain energy and excited to higher state B^*. Hence, A atoms lose energy and return to lower state

$$\boxed{A^* + B \rightarrow A + B^*}$$

Population inversion in the energy states of B is achieved. This method is used in He - Ne laser.

(8)

iii) Characteristics of Laser Light

The four Important characteristics

i) High directionality

ii) High Intensity

iii) Highly monochromatic

iv) Highly Coherent

i) High directionality

- Ordinary light source emits light in all directions.
- Laser source emits light only in one direction.
- Divergence of laser beam is very small.
- Laser Light has high directionality.

ii) High Intensity

- Laser source emits light as a narrow beam
- Energy is concentrated in a small region
- Concentration of energy gives a high intensity to the laser light.

iii) Highly monochromatic

- ordinary light spreads over a wavelength range of the order of 100 nm
- Laser beam has the order of 1 nm.

iv) Highly coherent

- The light emitted from a laser source consists of wave trains.
- Wave trains have same frequency, phase and direction.
- They are coherent.

(3) Explain the construction and working of Nd-YAG laser with neat diagram.

- It is a four level laser.
- Nd stands for Neodymium and YAG for Yttrium Aluminium Garnet ($Y_3Al_5O_{12}$)

Principle

- The active medium Nd-YAG rod is optically pumped by Krypton flash tube.
- The neodymium ions (Nd^{3+}) are raised to excited energy levels.
- During transition from metastable state to ground state a laser beam of wavelength $1.064\ \mu m$ is emitted.

Construction

- A small amount of yttrium ions (Y^{3+}) is replaced with neodymium ions (Nd^{3+}) in the active medium of Nd-YAG rod.
  - Nd-YAG Crystal is cut into a cylindrical rod.
  - The ends of this rod are highly polished and optically flat and parallel.
  - Krypton flash tube are kept in an elliptical reflector cavity in order to focus most of the light into Nd YAG rod.
  - Mirrors M_1 (total reflector) and M_2 (Partial reflector) act as a resonant cavity.
  - The power supply arrangement initiates the action of flash lamp.

Working

- When the flash lamp is energised, it gives out light radiation.
- Nd^{3+} ions get excited to higher energy levels by absorbing radiations of wavelengths 0.73 μm and 0.80 μm from the input light.
- By non-radiative transitions, these excited Nd^{3+} ions drop to metastable state E_2 and hence population inversion is achieved between the levels E_2 and E_3.
- Transition from the metastable state to a lower energy state leads to laser output of wavelength 1.064 m (infra red region).

Characteristics

Type : Four level laser

Active medium : Nd YAG rod

Pumping Method : optical pumping

Pumping Source : krypton flash tube

optical resonator : Two ends of Nd-YAG rod published with silver

Power output : 20 kW

Nature of output : Pulsed or Continuous beam of light

Wavelength of output : 1.06 μm (infra-red)

Advantages

- High Power output
- Easy to achieve population inversion
- Requires only lower threshold voltage.

Disadvantages

- Energy levels of Nd^{3+} ions are complicated.

Applications

- In long distance communication
- In remote sensing
- In engineering applications like cutting, welding, drilling etc
- In medical applications like endoscopy, dental surgery, urology, neurosurgery etc.

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(4) Explain the modes of vibrations of CO_2 molecule. Describe the construction and working of CO_2 laser with necessary diagrams.

- It is a four level molecular gas laser.
- Transition takes place between vibrational energy, states of carbondioxide molecules.
- Very efficient laser.

Energy states of CO_2 molecules:

Molecules vibrates in three independent modes. They are

i) Symmetric stretching mode

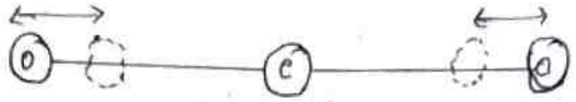
ii) Bending mode

iii) Asymmetric stretching mode,

i) Symmetric Stretching Mode

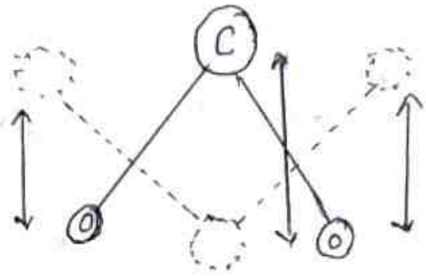
In this mode of vibration, Carbon atom is at rest. Both oxygen atoms vibrate such that they are moving either towards or away from the fixed Carbon atom against each other.

• Hence change in bond length occurs



ii) Bending mode

In this mode of vibration, both oxygen atoms and carbon atom vibrate perpendicular to molecular axis.



iii) Asymmetric stretching

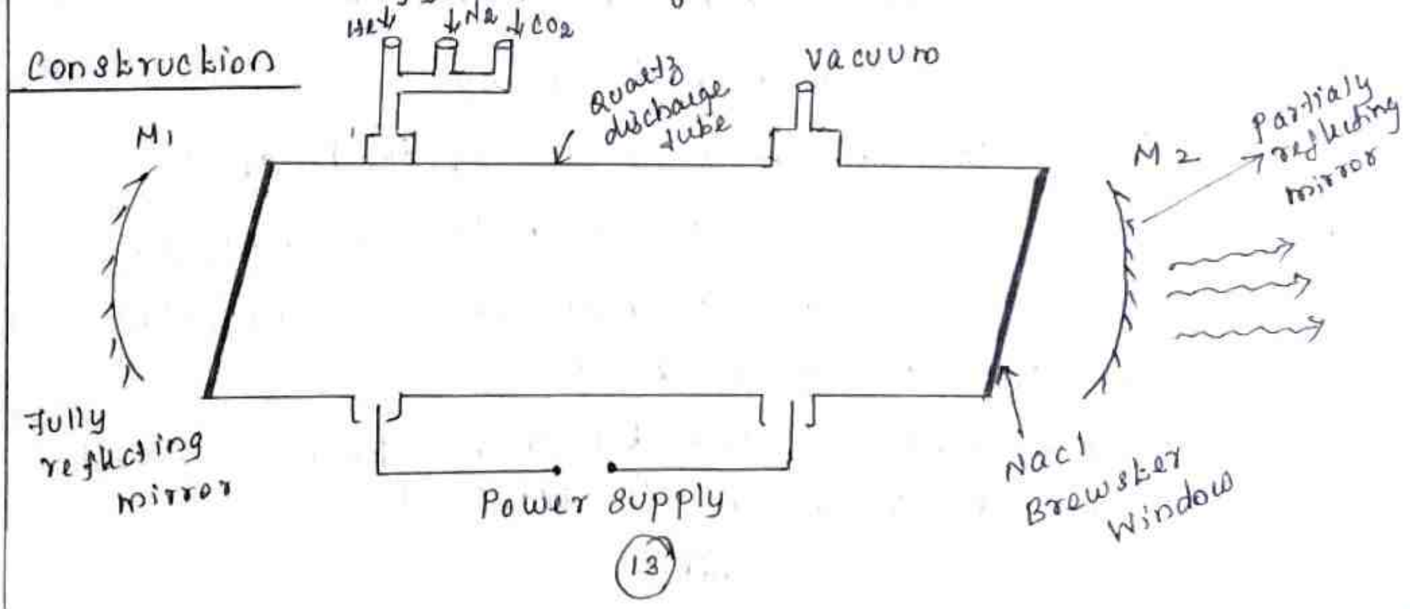
In this mode of vibration, both oxygen atoms and carbon atom vibrate asymmetrically.

If carbon atom moves in one direction, oxygen atoms move in opposite direction.

Principle

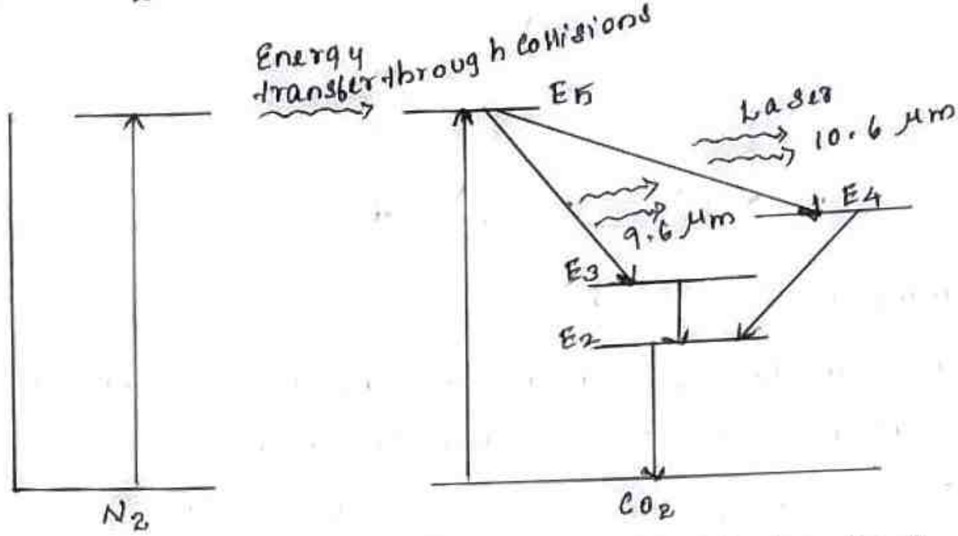
The laser transition takes place between the vibrational energy states of CO_2 molecules.

Construction



- It consists of a quartz discharge tube 5 m long and 2.5 cm in a diameter.
- The discharge tube is filled with the gas mixture of CO_2, nitrogen and helium with suitable pressures.
- Terminals connected to D.C. power supply.
- Ends of this tube are fitted with NaCl Brewster Windows.
- Laser Light generated.
- M_1 and M_2 are the mirrors.

Working



When the electrical discharge occurs in gas mixture, the electrons collide with nitrogen molecules and they are raised to excited energy states.

The process is represented by the equation

$$\boxed{N_2 + e^* \rightarrow N_2^* + e}$$

N_2 - Nitrogen molecule in ground state

e^* - Electron with high energy

N_2^* - Nitrogen molecule in excited state

e - Same electron with lesser energy.

- Excited energy level of nitrogen is very close to E_5 energy level of CO_2 molecules
- CO_2 molecules excited by energy transfer and population inversion is achieved.

The process is represented by the equation

$$N_2^* + CO_2 \longrightarrow CO_2^* + N_2$$

N_2^* - N_2 in excited state

CO_2 - CO_2 in ground state

CO_2^* - CO_2 in excited state

N_2 - N_2 in ground state

- Emitted photon triggers laser action in the tube

Two possible types of laser transition

i) Transition $E_5 - E_4$

Produces a laser beam of wavelength 10.6 μm.

ii) Transition $E_5 - E_3$

Produces a laser beam of wavelength 9.6 μm.

- Power output is 10 kW.

Characteristics

Type : Molecular gas and four level laser

Active medium : CO_2, N_2 and He

Pumping method : Electrical discharge method.

optical resonator : two concave mirrors

Power output : 10 kW

Nature of output : Continuous wave or pulsed wave

Wavelength of output : 9.6 μm and 10.6 μm

Advantages

- High power output
- Simple construction
- High efficiency
- Temperature can be reduced by decreasing the diameter of the tube.

(15)

Disadvantages

- Contamination of gases may occur which decreases laser efficiency.
- output efficiency depends on operating temperature.
- Corrosion may occur

Applications

- In material Processing like Welding, Cutting, drilling
- In remote Sensing
- In open air Communication
- In micro surgery, bloodless surgery, dental care,
- In liver and lung diseases.

(5) Explain the principle, construction and working of a semiconductor diode laser with necessary diagrams
[Homojunction and Heterojunction]

a) Semiconductor laser [Homojunction]

- Specially fabricated P-n junction diode.
- Emits forward bias when it is forward bias

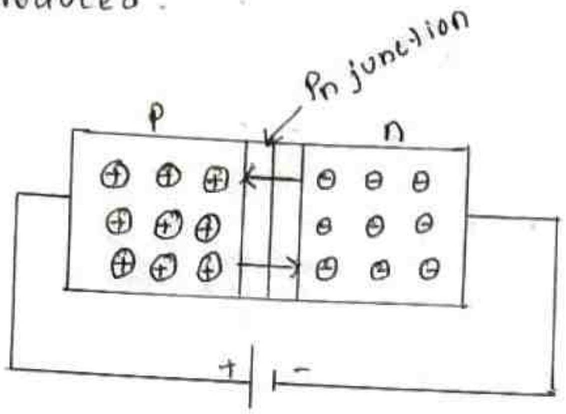
Principle

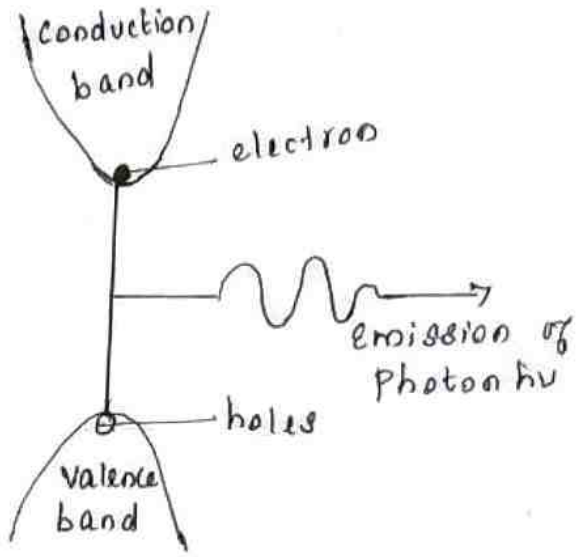
when the P-n junction diode is forward-biased the electrons from n-region and holes from P-region cross the junction and recombine with each other.

During the recombination process, the light radiation is released from direct band gap semiconductors like GaAs. This light radiation is known as recombination radiation

· During recombination stimulates other electrons and holes to recombine.

· Stimulated emission takes place and laser light is Produced.





Construction

· The diode is made from semiconductor material GaAs.

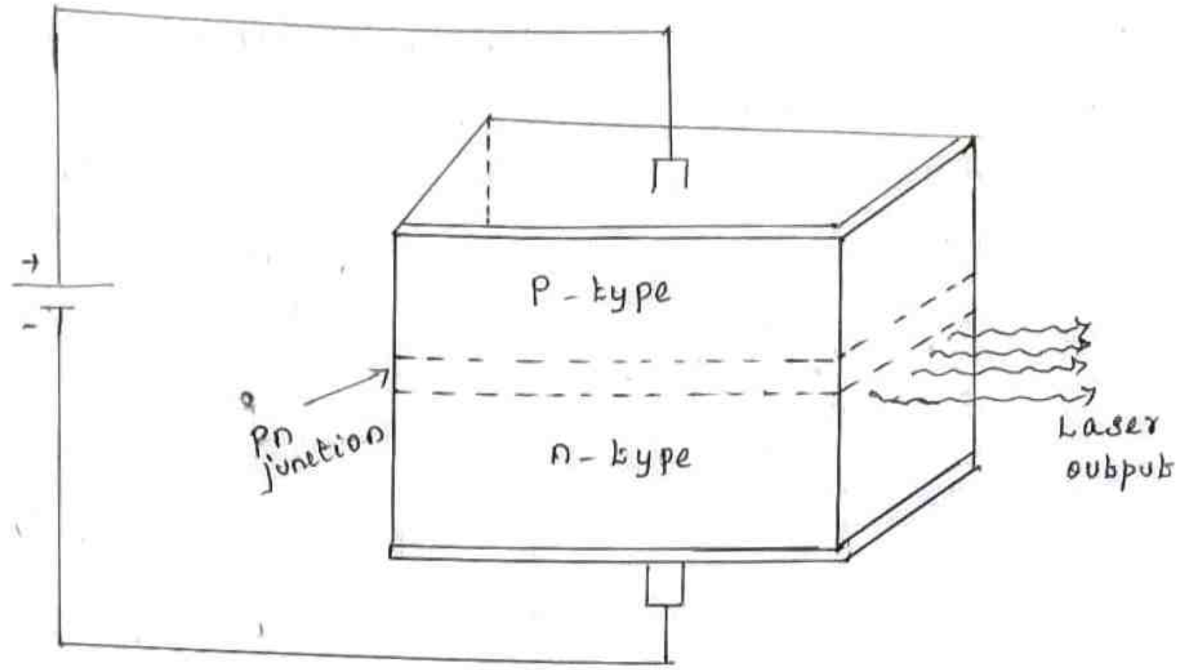
· Crystal is cut in the form of a Platelet having a thickness of 0.5 mm.

· The metal electrodes are connected to both upper and lower surfaces of semiconductor diode.

· Forward bias voltage is applied through metal electrodes.

· The sides of P and N through which the emitted light is coming out are well polished and Parallel to one another and they constitute the optical resonator system.

· The battery provides the necessary bias to the diode.



Working

- The pn junction is forward biased with large applied voltage.
- The electrons from n-side and holes from P-side are injected into the junction region in considerable Concentration.
- The excess minority electron in the P-region and excess minority holes in the n-region produce population inversion of minority charge carriers.
- The recombination of electrons and holes at the junction produces light emission.
- As voltage increased, stimulated recombination of photons occurs.

These photons moving at the plane of the junction travel back and forth by reflection between two polished surfaces of the junction.

- After gaining enough strength, laser beam of wavelength 8400 Å emitted from the junction

The wavelength of laser light is given by

$$E_g = h\nu = \frac{hc}{\lambda}$$

$$\boxed{\lambda = \frac{hc}{E_g}}$$

E_g - band gap energy

Characteristics

Type : Solid state semiconductor laser

Active medium : A pn junction diode made from a single crystal of gallium arsenide.

Pumping method : Direct conversion method

Power output : Few mW.

Nature of output : Continuous wave or pulsed output

Wavelength of output : 8300 Å to 8500 Å

Advantages

- High Power output
- High Coherence
- High Stability
- Continuous waveform

Disadvantages

- High Cost
- Fabrication is difficult
- Large divergence

Applications

- In optical communication
- Laser printing
- Data Storage
- Used as a pain killer
- Used to heal the wounds by infrared radiation.

b) Semiconductor laser [Hetero junction]

A diode laser with a pn junction made up of different semiconductor materials in two regions - n-type and p-type is known as heterojunction semiconductor laser.

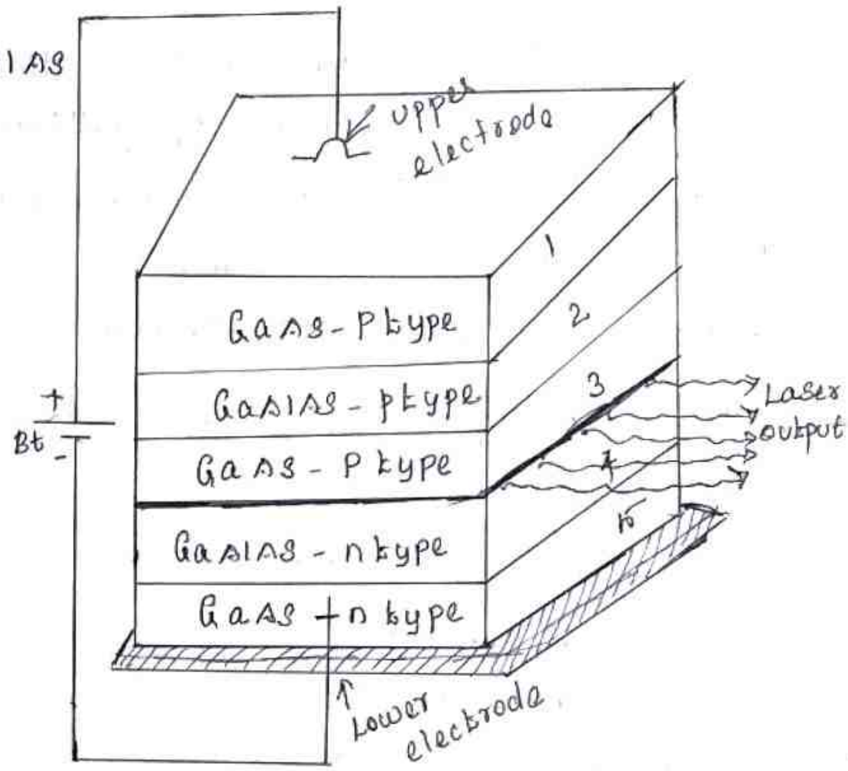
Principle

During recombination process, light photon is released.

Example

GaAs and GaAlAs

Construction



- A layer of GaAs p-type (3rd layer) having narrow band gap acts as an active region.
- This 3rd layer is sandwitched between the 2nd layer (GaAlAs - P-type) and 4th layer (GaAlAs - n-type)
- The end faces of junctions of 3rd layer and 4th layer are well polished and they act as an optical resonator.
- Proper biasing is used.

(20)

Working

- The diode is forward bias
- Charge carriers are produced in the band gap layer
- Charge carriers are injected continuously to the 3rd layer from 2nd and 4th layer to achieve population inversion condition.
- These injected carriers combine and produces spontaneously emitted photons.
- These photons stimulate the injected charge carriers to emit photons of same phase and wavelength.
- photons emerge out as an intense, coherent beam of laser of wavelength which lies in IR region.

Advantages

- Highly directional with almost zero divergence
- Threshold current density is drastically reduced
- The stripe geometry provides a good stability and larger lifetime.

Discuss the applications of Laser

i) This property can be used in material Processing like

- Industrial Cutting
- Industrial drilling
- Industrial Welding

ii) Lasers are used to scan the universal barcodes to identify products.

iii) used to take three - dimensional photography [Holography]

iv) Lasers find applications in isotope seperation, in fabricating microelectronic circuits etc.

v) In the field of chemistry, they are used to initiate chemical and photo chemical reactions.

vi) In the field of fiber optic communication, thousands of television programs and telephone conversations can be transmitted simultaneously using laser beam.

vii) To store and retrieve data in optical discs.

Medical applications

- In the treatment of detached retinas
- To perform microsurgery and bloodless operations to cure cancers and skin tumors.
- Nose, ear, throat surgery.
- To shatter kidney stones.
- To remove diseased body tissues.
- In the removal of tattoos or birthmarks etc.

Other applications

- As range finder in military application - LIDAR (Light Detection And Ranging)
- under water communication between submarines.
- To determine ozone concentration.
- To detect absolute rotation of earth.
- To measure the distance between earth and moon accurately
- objects like aeroplanes, missiles etc, can be destroyed in a few seconds by passing powerful laser beam on to them. So laser is called a <u>death ray</u> instrument.

Unit - IV

Basic Quantum Mechanics

Photons and light waves - Electrons and matter wave - Compton effect - The schrodinger eqn (Time dependent and time independent eqn) - meaning of wave function - Normalization - free particle - Particle in a infinite potential Well : 1D, 2D and 3D Boxes - Normalization, Probabilities and the Correspondence Principle.

① **Explain Debroglie hypothesis in electrons and matter waves and its properties**

According to de-broglie hypothesis, a moving particle is always associated with waves.

De-broglie waves and its wavelength

The waves associated with the matter particles are called matter waves or de-broglie waves.

From planck's theory

$$E = h\nu \longrightarrow (1)$$

According to Einstein's mass energy relation

$$E = mc^2 \longrightarrow (2)$$

m - mass of photon

c - Velocity of photon

Equating (1) and (2)

$$h\nu = mc^2$$

$$h\,c/\lambda = mc^2$$

$$\lambda = \frac{h\cancel{c}}{mc^{\cancel{2}}} \qquad \left[\nu = c/\lambda\right]$$

$$\lambda = h/mc \quad \text{(electromagnetic radiation)}$$

$mc = p$ - momentum of a photon

$$\lambda = h/p \longrightarrow (4)$$

$$\boxed{\lambda = \frac{h}{mv}} \longrightarrow (5) \qquad p = mv$$

Eqn (5) is known as de-Broglie's wave eqn.

De-broglie wavelength interms of energy

$$E = \frac{1}{2} mv^2$$

$$2E = mv^2 \longrightarrow (6)$$

Multiplying by 'm' on both sides

$$2mE = m^2 v^2$$
$$m^2 v^2 = 2mE$$
Taking Square root on both Sides
$$\sqrt{m^2 v^2} = \sqrt{2mE}$$
$$mv = \sqrt{2mE}$$
$$\lambda = \frac{h}{mv} \longrightarrow (7)$$
Substituting mv in eqn (7)
$$\boxed{\lambda = \frac{h}{\sqrt{2mE}}}$$

Debroglie's wavelength associated with electrons

Workdone on the electron = eV

Workdone = Kinetic energy
$$eV = \frac{1}{2} mv^2 \longrightarrow (8)$$
$$2eV = mv^2$$
multiply by m on both sides
$$2meV = m^2 v^2$$
$$m^2 v^2 = 2meV$$
Taking square root on both sides
$$\sqrt{m^2 v^2} = \sqrt{2meV}$$
$$mv = \sqrt{2meV} \longrightarrow (9)$$
$$\lambda = \frac{h}{mv} \longrightarrow (10)$$
substituting mv in (10)
$$\boxed{\lambda = \frac{h}{\sqrt{2meV}}} \longrightarrow (11)$$

Properties of matter waves -

i) If the mass of the particle is Smaller, then the wavelength associated with that particle is longer.

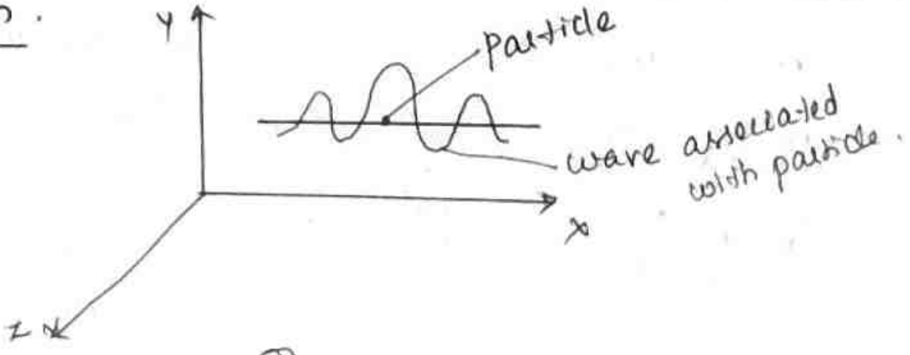
ii) If the velocity of the particle is small, then the wavelength associated with that particle is longer.

iii) If $v = 0$, then $\lambda = \infty$, i.e) the wave becomes interminable and if $v = \infty$, then $\lambda = 0$.

iv) waves do not depend on the charge of the particle.

v) The velocity of de-Broglie's waves is not constant since it depends on the velocity of the material particle.

(2) Derive an eqn for Schrodinger Time independent wave eqn and dependent eqn.



- Consider a wave associated with a moving particle.
- x, y, z - Coordinates
- ψ - wave function
- t - time

The classical differential eqn for wave motion is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \longrightarrow (1)$$

v - wave velocity

Eqn (1) written as

$$\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \longrightarrow (2)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplacian operator

The solution of eqn (2) gives ψ as a periodic variations of time t,

$$\psi(x, y, z, t) = \psi_0(x, y, z)e^{-i\omega t}$$

$$\psi = \psi_0 e^{-i\omega t} \longrightarrow (3)$$

Differentiating eqn (3) with respect to t,

$$\frac{\partial \psi}{\partial t} = -i\omega\, \psi_0 e^{-i\omega t}$$

Again differentiating with respect to t

$$\frac{\partial^2 \psi}{\partial t^2} = (-i\omega)(-i\omega)\,\psi_0 e^{-i\omega t}$$

$$\frac{\partial^2 \psi}{\partial t^2} = i^2 \omega^2 \psi_0 e^{-i\omega t}$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi \longrightarrow (4)$$

$$[\therefore i^2 = -1, \quad \psi = \psi_0 e^{-i\omega t}]$$

Substituting eqn (4) in eqn (2)

$$\nabla^2 \psi = -\frac{\omega^2}{v^2}\psi$$

$$\nabla^2 \psi + \frac{\omega^2}{v^2}\psi = 0 \longrightarrow (5)$$

$$\omega = 2\pi\nu = \frac{2\pi v}{\lambda}$$

$$\frac{\omega}{v} = \frac{2\pi}{\lambda} \longrightarrow (6)$$

$$\frac{\omega^2}{v^2} = \frac{4\pi^2}{\lambda^2} \longrightarrow (7)$$

Substituting eqn (7) in eqn (5)

$$\nabla^2 \psi + \frac{4\pi^2}{\lambda^2}\psi = 0 \longrightarrow (8)$$

Substituting λ value $\boxed{\lambda = h/mv}$ in eqn (8)

$$\nabla^2 \psi + \frac{4\pi^2}{\frac{h^2}{m^2v^2}}\cdot \psi = 0$$

$$\nabla^2 \psi + \frac{m^2v^2\, 4\pi^2}{h^2}\cdot \psi = 0 \longrightarrow (9)$$

Total Energy = Potential Energy + K.E

$$E = V + \tfrac{1}{2}mv^2$$

$$E - V = \tfrac{1}{2}mv^2$$

$$2(E - V) = mv^2$$

$$mv^2 = 2(E - V)$$

Multiply 'm' on both sides

$$m^2v^2 = 2m(E - V) \longrightarrow (10)$$

Substituting eqn (10) in eqn (9)

$$\nabla^2 \psi + \frac{4\pi^2}{h^2} \times 2m(E - V)\psi = 0$$

$$\boxed{\nabla^2 \psi + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0} \longrightarrow (11)$$

Eqn (11) is known as Schrodinger Time independent eqn

3

$\hbar = h/2\pi$, $\hbar^2 = h^2/2^2\pi^2$

$$\hbar^2 = \frac{h^2}{4\pi^2}$$

$\hbar$ is a reduced Planck's Constant. $\longrightarrow (12)$

The eqn (11) modified by Substituting $\hbar$

$$\nabla^2\psi + \frac{m}{\frac{h^2}{8\pi^2}}(E-V)\psi = 0$$

$$\nabla^2\psi + \frac{m}{\frac{h^2}{2\times 2^2\pi^2}}(E-V)\psi = 0$$

$$\nabla^2\psi + \frac{2m}{\frac{h^2}{4\pi^2}}(E-V)\psi = 0 \longrightarrow (13)$$

substituting eqn (12) in eqn (13)

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0 \longrightarrow (14)$$

(or)

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi} \longrightarrow (15)$$

Special case

Consider one dimensional motion moving along only x direction

$$\boxed{\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0} \longrightarrow (16)$$

ii) Schroedinger Time dependent eqn:

Schroedinger time dependent wave eqn is derived from Schroedinger time independent wave eqn.

The Solution of classical differential eqn of wave motion is given by

$$\psi(x,y,z,t) = \psi_0(x,y,z)e^{-i\omega t} \longrightarrow (1)$$

Differentiating eqn (1) with respect to time t,

$$\frac{\partial\psi}{\partial t} = -i\omega\psi_0 e^{-i\omega t} \longrightarrow (2)$$

$$\frac{\partial\psi}{\partial t} = -i(2\pi\nu)\psi_0 e^{-i\omega t} \quad (\because \omega = 2\pi\nu)$$

$$\frac{\partial\psi}{\partial t} = -2\pi i\nu\psi \longrightarrow (3)$$

$(\psi = \psi_0 e^{-i\omega t})$

$$\frac{\partial\psi}{\partial t} = -2\pi i\frac{E}{h}\psi \quad \left(E = h\nu,\ \nu = \frac{E}{h}\right)$$

$$\frac{\partial\psi}{\partial t} = -i\frac{E}{\frac{h}{2\pi}}\psi \quad (\hbar = h/2\pi)$$

$$\frac{\partial\psi}{\partial t} = -i\frac{E}{\hbar}\psi \longrightarrow (4)$$

Multiplying 'i' on both sides in eqn (4)

$$i\frac{\partial\psi}{\partial t} = -i^2\frac{E}{\hbar}\psi$$

$$i\frac{\partial\psi}{\partial t} = \frac{E}{\hbar}\psi \quad \boxed{i^2 = -1}$$

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi \longrightarrow (5)$$

Schroedinger time independent wave eqn is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

Substituting $E\psi$ from eqn(5)

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t} \longrightarrow (7)$$

(or) $H\psi = E\psi \longrightarrow (8)$

$H = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)$ Hamiltonian operator.

$E = i\hbar\frac{\partial}{\partial t}$ is energy operator.

Eqn (7) is called schroedinger time dependent wave eqn.

③ Discuss free particle Problem starting from schrodinger wave eqn eigen

- Consider electrons propagating freely in space in the +ve x direction and not acted upon by any force.
- Electrons are not acted upon by any force their potential energy V is zero

∴ Schroedinger eqn

$$\nabla^2\psi + \frac{8\pi^2 m}{h^2}(E-V)\psi = 0, \text{ i.e.}$$

reduces to

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}E\psi = 0 \longrightarrow (1)$$

Taking $\frac{8\pi^2 mE}{h^2} = k^2$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

The general solution of the above eqn is

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

A & B - Constants

- waves Propagates in x-direction

$$\psi(x,t) = Ae^{ikx}e^{-i\omega t}$$

The allowed energy ... form a continuum and given by

$$E = \frac{h^2k^2}{8\pi^2 m} \longrightarrow (2)$$

from eqn (2)

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad (\because p = \sqrt{2mE})$$

$$k = \frac{\sqrt{2mE}}{\hbar} = \frac{p}{\hbar} \qquad \left(\frac{p}{h} = \frac{1}{\lambda}\right)$$

$$= \frac{p}{h/2\pi} = \frac{2\pi p}{h}$$

$$\boxed{k = 2\pi/\lambda}$$

k - known as wave vector

- Seen from the relation (2)

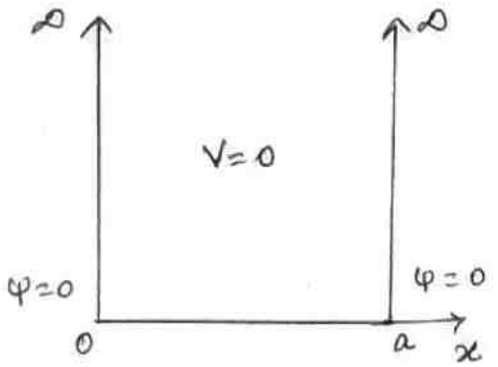
$$E \propto k^2$$

4 Derive an expression for a Particle in a Infinite Potential (One - Dimensional Box)

• Consider a particle of mass m moving between two rigid walls of potential well at $x = 0$ and $x = a$ along x-axis. The potential function is given by

$$\boxed{\begin{array}{l} V(x) = 0 \quad \text{for } 0 < x < a \\ V(x) = \infty \quad \text{for } 0 \geq x \geq a \end{array}}$$

This potential function is known as Square well potential



Schroedinger's wave eqn in one dimension is given by

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0 \longrightarrow (1)$$

$V = 0$ between the walls, eqn (1) reduces to

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 \longrightarrow (2)$$

Substituting $\frac{2mE}{\hbar^2} = k^2$ in eqn (2)

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \longrightarrow (3)$$

The general soln for eqn (3)

$$\psi(x) = A \sin kx + B \cos kx \longrightarrow (4)$$

A & B are two unknown Constants.

Boundary Conditions (i)

$\psi = 0$ at $x = 0$

Applying condition in eqn (4)

$$0 = A \sin 0 + B \cos 0 \qquad \left[\begin{array}{l} \sin 0 = 0 \\ \cos 0 = 1 \end{array}\right]$$

$$0 = 0 + B \times 1$$

$$B = 0.$$

Boundary Conditions (ii)

$\psi = 0$ at $x = a$

Applying condition in eqn (4)

$$0 = A \sin ka + 0 \qquad \boxed{B = 0}$$

$$A \sin ka = 0$$

$$A = 0 \quad \text{or} \quad \sin ka = 0$$

$$\sin ka = 0$$

$$ka = n\pi$$

$$k = \frac{n\pi}{a} \longrightarrow (5) \qquad n = 1, 2, 3 \ldots$$

on squaring eqn (5)

$$k^2 = \frac{n^2\pi^2}{a^2} \longrightarrow (6)$$

$$k^2 = \frac{2mE}{\hbar^2} = \frac{2mE}{\frac{h^2}{4\pi^2}} \qquad [\because \hbar = h/2\pi]$$

$$k^2 = \frac{4\pi^2\, 2mE}{h^2}$$

$$k^2 = \frac{8\pi^2 mE}{h^2} \longrightarrow (7)$$

6

Equating eqn (6) and (7)

$$\frac{n^2\pi^2}{a^2} = \frac{8\pi^2 m E}{h^2}$$

Energy of the Particle

$$E = \frac{n^2h^2}{8ma^2} \longrightarrow (8)$$

Substituting eqn (5) in (4)

$$\psi_n(x) = A \sin \frac{n\pi x}{a} \longrightarrow (9)$$

$n = 1, 2, 3$.

E_n - eigen value

ψ_n - eigen function.

Normalisation of wave function

Probability density is given by

$\psi^* \psi$

$$\psi_n(x) = A \sin \frac{n\pi x}{a}$$

$$\psi^*\psi = A \sin \frac{n\pi x}{a} \times A \sin \frac{n\pi x}{a}$$

[$\psi = \psi^*$ the wave fn is real]

$$\psi^*\psi = A^2 \sin^2 \left[\frac{n\pi x}{a}\right] \longrightarrow (10)$$

$$\int_0^a \psi^* \psi \, dx = 1 \longrightarrow (11)$$

Substituting $\psi^*\psi$ from eqn (10) in eqn (11)

$$\int_0^a A^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = 1$$

$$A^2 \int_0^a \frac{\left(1 - \cos\left(\frac{2\pi n x}{a}\right)\right)}{2} dx = 1 \qquad \left(\sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$\frac{A^2}{2}\left[\int_0^a dx - \int_0^a \cos\left(\frac{2\pi n x}{a}\right) dx\right] = 1$$

$$\frac{A^2}{2}\left[(x)_0^a - \left[\frac{\sin\left(\frac{2\pi n x}{a}\right)}{\frac{2\pi n}{a}}\right]\right]_0^a = 1$$

The second term of the integral becomes zero at both limits.

$$\frac{A^2}{2}(x)_0^a = 1$$

$$\frac{A^2 a}{2} = 1$$

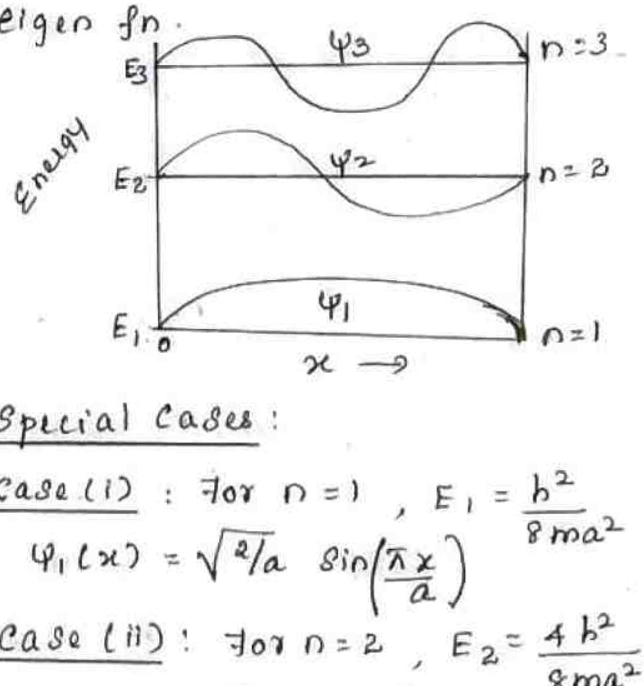
$$A^2 = 2/a$$

$$\boxed{A = \sqrt{2/a}} \longrightarrow (12)$$

Substituting eqn (12) in (9)

$$\boxed{\psi_n = \sqrt{2/a}\, \sin \frac{n\pi x}{a}} \longrightarrow (13)$$

Eqn (13) known as normalised eigen fn.



Special Cases:

Case (i) : For $n = 1$, $E_1 = \frac{h^2}{8ma^2}$

$$\psi_1(x) = \sqrt{2/a}\, \sin\left(\frac{\pi x}{a}\right)$$

Case (ii) : For $n = 2$, $E_2 = \frac{4h^2}{8ma^2} = 4E_1$

$$\psi_2(x) = \sqrt{2/a}\, \sin\left(\frac{2\pi x}{a}\right)$$

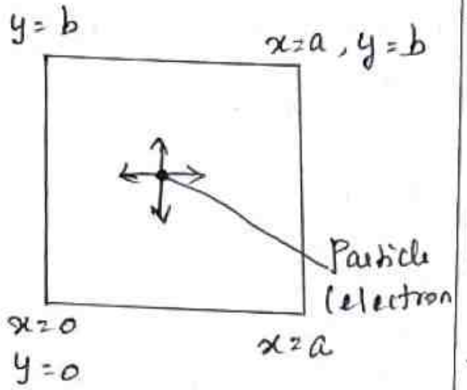
Case (iii) : For $n = 3$, $E_3 = \frac{9h^2}{8ma^2} = 9E_1$

$$\psi_3(x) = \sqrt{2/a}\, \sin\left(\frac{3\pi x}{a}\right)$$

(7)

(5) Derive an expression for two Dimensions potential well (2D)

The solution of one-dimensional potential well is extended for a two dimensional potential well.



Energy of the particle

$$E = E_{nx} + E_{ny}$$

$$E_{n_x n_y} = \frac{n_x^2 h^2}{8ma^2} + \frac{n_y^2 h^2}{8ma^2}$$

If $a = b$

$$\boxed{E_{n_x n_y} = \frac{h^2}{8m}\left[\frac{n_x^2}{a^2} + \frac{n_y^2}{a^2}\right]}$$

Two dimensional well is written as

$$\psi_{n_x n_y} = \sqrt{2/a}\ \sin\left(\frac{n_x \pi x}{a}\right) \times \sqrt{2/b}\ \sin\left(\frac{n_y \pi y}{b}\right)$$

$$\psi_{n_x n_y} = \sqrt{2/a} \times \sqrt{2/b}\ \sin\frac{n_x \pi x}{a}\ \sin\left(\frac{n_y \pi y}{b}\right)$$

$$\boxed{\psi_{n_x n_y} = \sqrt{\frac{4}{ab}}\left(\sin\frac{n_x \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{b}\right)} \longrightarrow (2)$$

Example

$n_x = 1,\ n_y = 2$

$n_x^2 + n_y^2 = 1^2 + 2^2 = 1 + 4 = 5$

For a combination $n_x = 2,\ n_y = 1$

$n_x^2 + n_y^2 = 2^2 + 1^2 = 4 + 1 = 5$

$$\boxed{E_{12} = E_{21} = \frac{5h^2}{8ma^2}} \longrightarrow (3)$$

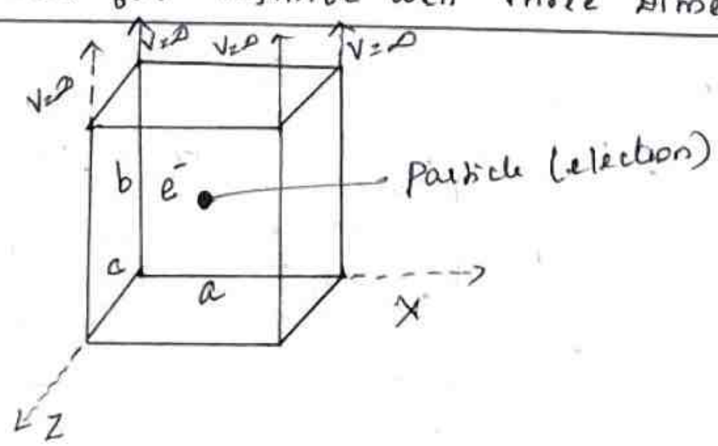
The corresponding wave fn

$$\boxed{\psi_{12} = \sqrt{4/ab}\ \sin\frac{\pi x}{a}\ \sin\frac{2\pi y}{b}}$$

$$\boxed{\psi_{21} = \sqrt{\frac{4}{ab}}\ \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{y}{b}\right)}$$

$\longrightarrow (4)$

(6) Derive an expression for infinite well Three Dimensions (3 D Box)



- The solution of one dimensional potential well is extended for a three dimensional (3D) potential box.
- In a three dimensional potential box, the particle can move in any direction in space.

Energy of the particle

$$= E_x + E_y + E_z$$

$$E_{n_x n_y n_z} = \frac{n_x^2 h^2}{8ma^2} + \frac{n_y^2 h^2}{8mb^2} + \frac{n_z^2 h^2}{8mc^2}$$

If $a = b = c$ for a cubical box

$$E_{n_x n_y n_z} = \frac{h^2}{8m}\left[\frac{n_x^2}{a^2} + \frac{n_y^2}{a^2} + \frac{n_z^2}{a^2}\right]$$

$$E_{n_x n_y n_z} = \frac{h^2}{8ma^2}\left[n_x^2 + n_y^2 + n_z^2\right] \rightarrow (1)$$

The corresponding normalised wave fn of the particle in the three dimension well is written as

$$\psi_{n_x n_y n_z} = \sqrt{\frac{2}{a}}\sin\left(\frac{n_x \pi x}{a}\right) \cdot \sqrt{2/b}\,\sin\left(\frac{n_y \pi y}{b}\right) \cdot \sqrt{2/c}\,\sin\left(\frac{n_z \pi z}{c}\right)$$

$$\psi_{n_x n_y n_z} = \sqrt{\frac{8}{abc}}\sin\left(\frac{n_x \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{b}\right)\sin\left(\frac{n_z \pi z}{c}\right) \rightarrow (2)$$

Example

Suppose a state has quantum numbers

$n_x = 1$, $n_y = 1$, $n_z = 2$

$$n_x^2 + n_y^2 + n_z^2 = 1^2 + 1^2 + 2^2$$

$$= 1 + 1 + 4 = 6$$

Similarly for a combination $n_x = 1$, $n_y = 2$, $n_z = 1$ and for combination $n_x = 2$, $n_y = 1$, $n_z = 1$

$$n_x^2 + n_y^2 + n_z^2 = 1^2 + 1^2 + 2^2 = 1 + 1 + 4$$

$$= 6$$

$$\therefore E_{112} = E_{121} = E_{211} = \frac{6h^2}{8ma^2} \rightarrow (3)$$

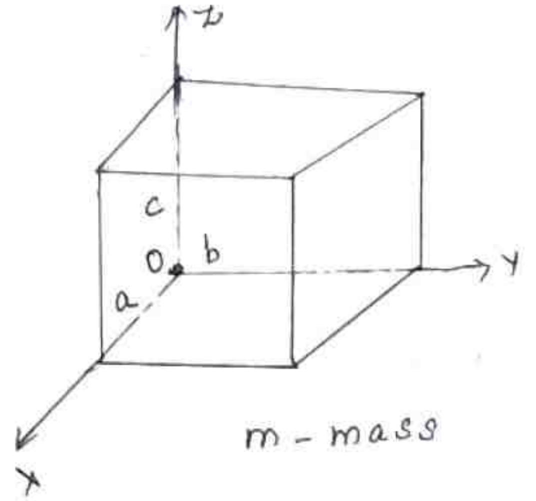
The corresponding wave functions is written as

$$\psi_{112} = \sqrt{\frac{8}{a^3}}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{2\pi z}{c}$$

$$\psi_{121} = \sqrt{\frac{8}{a^3}}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b}\sin\frac{\pi z}{c}$$

$$\psi_{211} = \sqrt{\frac{8}{a^3}}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{\pi z}{c}$$

(6) Derive an expression for particle in a Rectangular Three-dimensional Infinite well



If there is no force acting on the particle inside the box the region

$$0 < x < a$$
$$0 < y < b$$
$$0 < z < c$$

Potential energy $V(x,y,z) = 0$

outside box $V(x,y,z) = \infty$

wave eqn of the particle

The schroedinger time independent wave eqn is

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \longrightarrow (1)$$

(or) $$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2m}{\hbar^2} E\psi = 0 \longrightarrow (2)$$

$\psi(x,y,z)$ is equal to the product of three functions X, Y, Z

$$\psi(x,y,z) = X(x)\,Y(y)\,Z(z) \longrightarrow (3)$$

Substituting this eqn in (2)

$$YZ\frac{d^2X}{dx^2} + ZX\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} + \frac{2mE}{\hbar^2}XYZ = 0 \longrightarrow (4)$$

Dividing eqn (4) by XYZ

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{2mE}{\hbar^2} = 0. \longrightarrow (5)$$

In this eqn $\frac{2mE}{\hbar^2}$ is a constant for a particular value of the K.E

$$E = E_x + E_y + E_z \longrightarrow (6)$$

From eqns (5) & (6)

$$\left[\frac{1}{X}\frac{d^2X}{dx^2} + \frac{2mE_x}{\hbar^2}\right] + \left[\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{2mE_y}{\hbar^2}\right] + \left[\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{2mE_z}{\hbar^2}\right] = 0$$

Three independent eqns

$$\frac{d^2X}{dx^2} + \frac{2mE_x}{\hbar^2}X = 0 \longrightarrow (7)$$

$$\frac{d^2Y}{dy^2} + \frac{2mE_y}{\hbar^2}Y = 0 \longrightarrow (8)$$

$$\frac{d^2Z}{dz^2} + \frac{2mE_z}{\hbar^2}Z = 0 \longrightarrow (9)$$

Eqn (7) is the eqn for one dimensional case

The boundary condition applicable to the solution is

$X(0) = X(a) = 0$

So the eigen values of E_x

$$E_x = \frac{\pi^2 \hbar^2}{2ma^2} n_x^2 \longrightarrow (10)$$

$n_x = 1, 2, 3 \ldots$

the corresponding normalized eigen functions

$$X(x) = \sqrt{2/a}\, \sin \frac{n_x \pi x}{a} \longrightarrow (11)$$

The solution for Y and Z of the same form

$$E_y = \frac{\pi^2 \hbar^2}{2mb^2} n_y^2 \longrightarrow (12)$$

$$Y(y) = \sqrt{2/b}\, \sin \frac{n_y \pi y}{b} \longrightarrow (13)$$

$$E_z = \frac{\pi^2 \hbar^2}{2mc^2} n_z^2 \longrightarrow (14)$$

$$Z(z) = \sqrt{2/c}\, \sin \frac{n_z \pi z}{c} \longrightarrow (15)$$

Eigen values of energy substituting the expressions for E_x, E_y and E_z

$$E_{n_x}, E_{n_y}, E_{n_z} = \frac{\pi^2 \hbar^2}{2m} \left[\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_c^2}{c^2} \right] \longrightarrow (16)$$

$n_x = 1, 2, 3 \ldots$

$n_y = 1, 2, 3 \ldots$

$n_z = 1, 2, 3 \ldots$

These values are called the energy levels of the particle.

wave function

$$\psi_{n_x, n_y, n_z}(x, y, z) = X(x)\, Y(y)\, Z(z)$$

$$= \sqrt{\frac{8}{abc}}\, \sin \frac{n_x \pi x}{a} \cdot \sin \frac{n_y \pi y}{b} \cdot \sin \frac{n_z \pi z}{c} \longrightarrow (17)$$

n_x, n_y, n_z - integers

The wave function is zero outside the box. It is easily proved the wave function normalized

$$\frac{8}{abc} \int_0^a \left(\sin \frac{n_x \pi x}{a} \right)^2 dx \int_0^b \left(\sin \frac{n_y \pi y}{b} \right)^2 dy \int_0^c \sin \left(\frac{n_z \pi z}{c} \right)^2 dz = 1$$

(7) State and prove Bohr's correspondance principle

Statement

The principle states that for large quantum numbers quantum physics gives the same results as those of classical physics.

Proof:

The velocity of an electron revolving round the nucleus in an orbit of radius r is given by

$$v^2 = \frac{ke^2}{mr} \longrightarrow (1)$$

Taking root on both sides

$$\sqrt{v^2} = \sqrt{\frac{ke^2}{mr}} = \frac{\sqrt{k}\sqrt{e^2}}{\sqrt{mr}}$$

$$v = e\sqrt{\frac{k}{mr}} = \frac{e}{\sqrt{4\pi\varepsilon_0 mr}} \longrightarrow (2)$$

$$r = \frac{n^2h^2}{4\pi^2 mkze^2}$$

$$r = \frac{n^2h^2}{\frac{4\pi^2 m \times 1}{4\pi\varepsilon_0} \times 1 \times e^2} = \frac{n^2h^2\varepsilon_0}{\pi me^2} \longrightarrow (3)$$

(putting $z=1$ for hydrogen $k = \frac{1}{4\pi\varepsilon_0}$)

The frequency of revolution

$$\nu = \frac{v}{2\pi r} \longrightarrow (4)$$

Substituting for v we have

$$\nu = \frac{1}{2\pi}\frac{e}{\sqrt{4\pi\varepsilon_0 mr}\,r} = \frac{1}{2\pi}\frac{e}{(4\pi\varepsilon_0 mr)^{1/2} r^1}$$

$$= \frac{1}{2\pi}\frac{e}{(4\pi\varepsilon_0 m)^{1/2} r^{1/2} r^1}$$

$$= \frac{1}{2\pi}\frac{e}{(4\pi\varepsilon_0 m)^{1/2} r^{3/2}}$$

Substituting for r

$$= \frac{1}{2\pi}\cdot\frac{e}{(4\pi\varepsilon_0 m)^{1/2}\left(\frac{n^2h^2\varepsilon_0}{\pi me^2}\right)^{3/2}}$$

$$\nu = \frac{me^4}{4\varepsilon_0^2h^3}\cdot\frac{1}{n^3} \longrightarrow (5)$$

According to Bohr's theory of the hydrogen atom

$$\frac{1}{\lambda} = R\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

$$= \frac{me^4}{8\varepsilon_0^2h^3}\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

$$\nu = \frac{c}{\lambda} = \frac{me^4}{8\varepsilon_0^2h^3}\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right] \longrightarrow (6)$$

when the quantum number involved are large

$n_1 = n$, $n_2 = n+1$, where $n >> 1$

$$\nu = \frac{me^4}{8\varepsilon_0^2h^3}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

$$= \frac{me^4}{8\varepsilon_0^2h^3}\left[\frac{(n+1)^2 - n^2}{n^2(n+1)^2}\right]$$

$$= \frac{me^4}{8\varepsilon_0^2h^2}\left[\frac{n^2+1^2+2n-n^2}{n^2(n+1)^2}\right]$$

$$= \frac{me^4}{8\varepsilon_0^2h^3}\left[\frac{2n+1}{n^2(n+1)^2}\right]$$

As $n >> 1$, so neglecting 1 as compared to n and $2n$

$$\nu = \frac{me^4}{8\varepsilon_0^2h^3}\cdot\frac{2}{n^3} \longrightarrow (7)$$

$$\nu = \frac{me^4}{4\varepsilon_0^2h^3} \longrightarrow (8)$$

The greater the quantum numbers, the closer quantum physics approaches the classical physics.

8. Explain compton effect and derive an expression for the wavelength of scattered photon. Also briefly explain its experimental verification

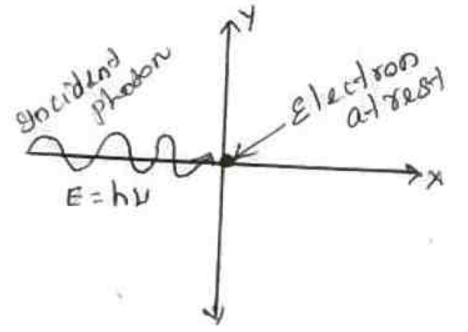
Statement:

When a beam of x-rays is scattered by a substance of low atomic number, the scattered x-ray radiation consists of two components.

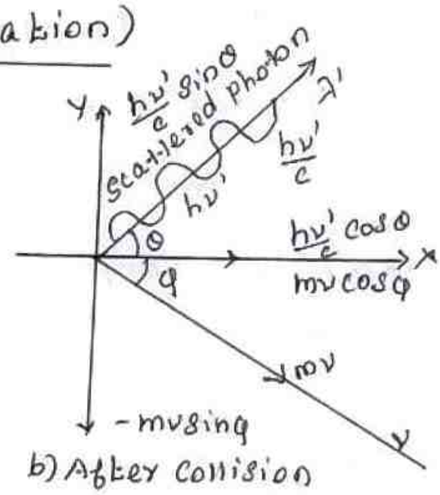
One component has the same wavelength λ as the incident ray and the other component has a slightly longer wavelength λ'.

The change in the wavelength of scattered x-rays is known as Compton shift. The phenomenon is called Compton effect.

Theory of Compton effect (Derivation)



a) Before collision



b) After Collision

Total Energy before Collision

Energy of incident photon = $h\nu$

Energy of electron at rest = m_0c^2

where

m_0 - rest mass of the electron

c - velocity of light

Total enery before Collision

$= h\nu + m_0c^2$

Total Energy after Collision

Energy of scattered photon = $h\nu'$

Energy of scattered electron = mc^2

Total energy after collision = $h\nu' + mc^2$

Applying the law of conservation of energy

Total energy before collision = Total energy after collision

$$h\nu + m_0c^2 = h\nu' + mc^2$$

$$mc^2 = h\nu - h\nu' + m_0c^2$$

$$mc^2 = h(\nu - \nu') + m_0c^2 \quad \rightarrow (1)$$

Total momentum along x axis

Momentum of photon along x axis $= h\nu/c$

Momentum of electron along x axis $= 0$

Total momentum along x axis $= h\nu/c$

After collision

Momentum of photon along x-axis
$= \frac{h\nu'}{c} \cos\theta$

Momentum of electron along x axis $= mv\cos\varphi$

Total momentum along x-axis after collision $= \frac{h\nu'}{c}\cos\theta + mv\cos\varphi$

Applying the law of Conservation of momentum.

Total momentum before collision = Total momentum after collision

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + mv\cos\varphi \longrightarrow (2)$$

$$\frac{h\nu}{c} - \frac{h\nu'}{c}\cos\theta = mv\cos\varphi$$

$$\frac{h}{c}(\nu - \nu'\cos\theta) = mv\cos\varphi$$

$$h(\nu - \nu'\cos\theta) = mvc\cos\varphi$$

$$mvc\cos\varphi = h(\nu - \nu'\cos\theta) \longrightarrow (3)$$

Total momentum along Y-axis

Before collision

Momentum of photon along Y-axis $= 0$

Momentum of electron along Y axis $= 0$

Total momentum along Y axis $= 0$

After collision

Momentum of photon along Y-axis
$= \frac{h\nu'}{c}\sin\theta$

Momentum of electrong along Y-axis $= -mv\sin\varphi$

[negative sign indicates negative Y-direction]

Total momentum along Y-axis

$= \frac{h\nu'}{c}\sin\theta - mv\sin\varphi$

Applying the law of conservation of momentum

Total momentum before collision = Total momentum after collision

$$0 = \frac{h\nu'}{c}\sin\theta - mv\sin\varphi$$

$$mv\sin\varphi = \frac{h\nu'}{c}\sin\theta \longrightarrow (4)$$

$$mvc\sin\varphi = h\nu'\sin\theta \longrightarrow (5)$$

squaring eqn(3) and eqn(5), and then adding, we get

$$(mvc\cos\varphi)^2 + (mvc\sin\varphi)^2 = h^2(\nu-\nu'\cos\theta)^2 + (h\nu'\sin\theta)^2 \longrightarrow (6)$$

L.H.S. of eqn (6)

$$= m^2v^2c^2 \qquad [\because \sin^2\varphi + \cos^2\varphi = 1]$$

R.H.S of eqn (6)

$$= h^2(\nu^2 - 2\nu\nu'\cos\theta + \nu'^2)$$

L.H.S = R.H.S of eqn (6) $[\because \sin^2\theta + \cos^2\theta = 1]$

$$m^2v^2c^2 = h^2(\nu^2 - 2\nu\nu'\cos\theta + \nu'^2) \longrightarrow (7)$$

squaring eqn (1) on both side,

$$(mc^2)^2 = (h(\nu-\nu') + m_0c^2)^2 \longrightarrow (8)$$

$$m^2c^4 = h^2(\nu^2 - 2\nu\nu' + \nu'^2) + 2h(\nu-\nu')m_0c^2 + m_0^2c^4 \longrightarrow (9)$$

subtracting eqn (7) from eqn (9)

$$m^2c^2(c^2 - v^2) = -2h^2\nu\nu'(1-\cos\theta) + 2h(\nu-\nu')m_0c^2 + m_0^2c^4 \longrightarrow (10)$$

From the theory of relativity

$$m = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \longrightarrow (11)$$

Squaring the eqn (11) on both sides

$$m^2 = \frac{m_0^2}{1-v^2/c^2} = \frac{m_0^2}{\frac{c^2-v^2}{c^2}} = \frac{m_0^2c^2}{c^2-v^2}$$

$$m^2(c^2-v^2) = m_0^2c^2$$

Multiplying c^2 on both sides

$$m^2c^2(c^2-v^2) = m_0^2c^2c^2 \rightarrow (12)$$

Substituting eqn (12) in eqn (10)

$$m_0^2c^4 = -2h^2vv'(1-\cos\theta) + 2h(v-v')m_0c^2 + m_0^2c^4$$

$$2h(v-v')m_0c^2 = 2h^2vv'(1-\cos\theta)$$

(or) $$\frac{v-v'}{vv'} = \frac{h}{m_0c^2}(1-\cos\theta)$$

$$\frac{v}{vv'} - \frac{v'}{vv'} = \frac{h}{m_0c^2}(1-\cos\theta) \rightarrow (13)$$

Multiplying c on both sides of eqn (13)

$$\frac{c}{v'} - \frac{c}{v} = \frac{hc}{m_0c^2}(1-\cos\theta)$$

$$\frac{c}{v'} - \frac{c}{v} = \frac{h}{m_0c}(1-\cos\theta)$$

$$\lambda' - \lambda = \frac{h}{m_0c}(1-\cos\theta) \quad \left[\because \frac{c}{v}=\lambda,\ \frac{c}{v'}=\lambda'\right]$$

Change in wavelength is given by

$$d\lambda = \frac{h}{m_0c}(1-\cos\theta) \rightarrow (14)$$

Case - 1, when $\theta = 0$

$$d\lambda = \frac{h}{m_0c}(1-\cos\theta) \quad [\cos\theta = 1]$$

$$d\lambda = \frac{h}{m_0c}(1-1)$$

$$= \frac{h}{m_0c} \times 0$$

$$\boxed{d\lambda = 0}$$

Case - 2

when $\theta = 90^\circ$

$$d\lambda = \frac{h}{m_0c}(1-\cos 90^\circ) \quad [\because \cos 90^\circ = 0]$$

$$d\lambda = \frac{h}{m_0c}$$

Substituting for h, m_0 and c

$$d\lambda = \frac{6.625\times10^{-34}}{9.11\times10^{-31}\times3\times10^8}$$

$$\boxed{d\lambda = 0.0243\,Å}$$

Case - 3 when $\theta = 180^\circ$.

$$d\lambda = \frac{h}{m_0c}(1-\cos 180^\circ)$$

$$d\lambda = \frac{h}{m_0c}(1-(-1)) \quad [\because \cos 180^\circ = -1]$$

$$d\lambda = \frac{h}{m_0c}(1+1)$$

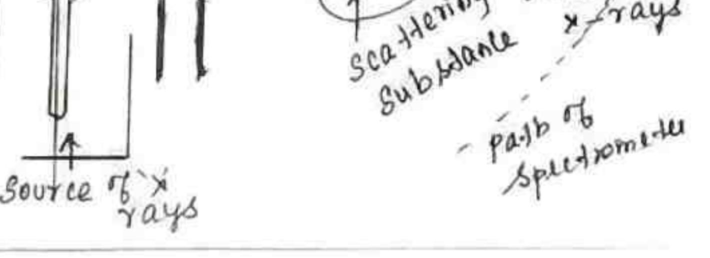
$$d\lambda = \frac{2h}{m_0c}$$

$$d\lambda = 2\times0.0243\,Å \quad \left[\because \frac{h}{m_0c} = 0.0243\,Å\right]$$

$$\boxed{d\lambda = 0.0486\,Å}$$

Change in wavelength is maximum at $\theta = 180^\circ$

ii) Experimental verification of Compton effect



• A beam of monochromatic x-rays of wavelength λ is made to incident on a scattering substance.
• The scattered x-rays are received by Bragg Spectrometer.

$$\boxed{\lambda' - \lambda = d\lambda = \frac{h}{m_0 c}(1 - \cos\theta)}$$

Unit : V

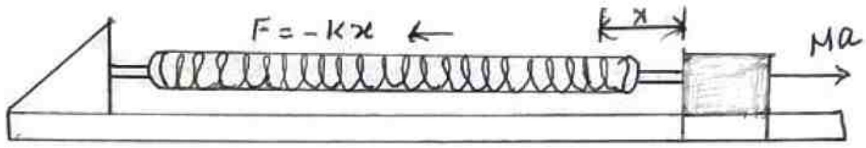
Applied Quantum Mechanics

The harmonic oscillator (qualitative) – Barrier penetration and quantum tunneling (qualitative) – Tunneling microscope – Resonant diode – Finite potential wells (qualitative) – Bloch's theorem for particles in a periodic potential – Basics of Kroning-Penney model and origin of energy bands.

① Discuss about Harmonic oscillator (Qualitative)

Definition

A particle undergoing simple harmonic motion is called a harmonic oscillator



Examples : Simple pendulum, an object floating in a liquid

If applied force moves the Particle through x, then restoring force F is given by

$$F \propto -x$$

$$F = -kx \longrightarrow (1)$$

The potential energy of the oscillator is

$$V = -\int F\,dx$$

$$V = k\int x\,dx = \tfrac{1}{2}kx^2$$

$$V = \tfrac{1}{2}kx^2 \longrightarrow (2)$$

k – force constant

In harmonic oscillator, angular frequency is given by

$$\omega = \sqrt{k/m}$$

Squaring on both sides

$$\omega^2 = \left(\sqrt{k/m}\right)^2, \quad \omega^2 = k/m, \quad k = m\omega^2$$

m – mass of the particle

Substituting k in eqn (1)

$$V = \tfrac{1}{2}m\omega^2 x^2 \longrightarrow (3)$$

wave eqns for the oscillator

The time independent Schrodinger wave eqn for linear motion of a Particle along the x-axis

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0 \longrightarrow (4)$$

Substituting V in eqn (4)

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \tfrac{1}{2}m\omega^2x^2\right)\psi = 0 \longrightarrow (5)$$

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2} - \frac{2m}{\hbar^2} \times \frac{1}{2}m\omega^2x^2\psi = 0$$

$$\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}x^2\right)\psi = 0 \longrightarrow (6)$$

This eqn is for the oscillator

①

Simplification of the wave eqn

$$y = ax \rightarrow (7)$$

$x = y/a$ where $a = \sqrt{\frac{m\omega}{\hbar}}$

$$\frac{d\psi}{dx} = \frac{d\psi}{dy}\frac{dy}{dx} = \frac{d\psi}{dy} a \qquad \left[\begin{array}{l} y = ax \\ dy = a\,dx \\ \frac{dy}{dx} = a \end{array} \right]$$

Differentiating

$$\frac{d^2\psi}{dx^2} = \frac{d^2\psi}{dy^2}\frac{d^2y}{dx^2}$$

$$\frac{d^2\psi}{dx^2} = \frac{d^2\psi}{dy^2} a^2 \qquad \left(\frac{d^2y}{dx^2} = a^2 \right)$$

$$\frac{d^2\psi}{dx^2} = a^2 \frac{d^2\psi}{dy^2} \rightarrow (8)$$

Substituting for $\frac{d^2\psi}{dx^2}$ and x^2 in eqn (6)

$$a^2 \frac{d^2\psi}{dy^2} + \left(\frac{2mE}{\hbar^2} - a^4 \frac{y^2}{a^2} \right)\psi = 0$$

$$a^2 \frac{d^2\psi}{dy^2} + \left(\frac{2mE}{\hbar^2} - a^2y^2 \right)\psi = 0 \qquad \left[\begin{array}{l} \therefore x = y/a \\ a = \sqrt{\frac{m\omega}{\hbar^2}} \\ a^2 = \frac{m\omega}{\hbar^2} \\ a^2 = \frac{m^2\omega^2}{\hbar^2} \\ a^4 = \frac{m^2\omega^2}{\hbar^2} \end{array} \right]$$

Dividing through out by a^2

$$\frac{d^2\psi}{dy^2} + \left(\frac{2mE}{a^2\hbar^2} - y^2 \right)\psi = 0 \rightarrow (9)$$

Substituting for a^2

$$\frac{d^2\psi}{dy^2} + \left[\frac{2mE}{\frac{m\omega}{\hbar}.\hbar^2} - y^2 \right]\psi = 0 \rightarrow (10)$$

(or) $$\frac{d^2\psi}{dy^2} + \left[\frac{2E}{\hbar\omega} - y^2 \right]\psi = 0$$

(or) $$\boxed{\frac{d^2\psi}{dy^2} + (\lambda - y^2)\psi = 0} \rightarrow (11)$$

where $\lambda = \frac{2E}{\hbar\omega}$.

②

Eigen values of the total energy E_n

The wave eqn for the oscillator is satisfied only for discrete values of total energies given by

$$\frac{2E}{\hbar\omega} = (2n+1) \quad \text{(or)}$$

$$E_n = \frac{1}{2}(2n+1)\hbar\omega$$

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega} \rightarrow (12)$$

$$E_n = (n + 1/2)\frac{h}{\cancel{2\pi}}\cancel{2\pi}\nu \qquad \begin{array}{l} \hbar = h/2\pi \\ \omega = 2\pi\nu \end{array}$$

$$E_n = \left(n + \frac{1}{2}\right)h\nu \rightarrow (13)$$

$n = 0, 1, 2, \ldots$

ν - frequency of the classical harmonic oscillator.

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{K/m} \qquad \left(\therefore \omega = \sqrt{K/m} \right)$$

From eqn (13)

i) Putting $n = 0$ in eqns (12) and (13)

$$E_0 = \frac{1}{2}\hbar\omega = 1/2\, h\nu \rightarrow (14)$$

This is called the ground state energy or the zero point vibrational energy of the harmonic oscillator.

$$E_n = (2n+1)E_0 \rightarrow (15)$$

ii) The eigen values of the total energy depend only on one quantum number n.

wave functions of the harmonic oscillator

$$\lambda = \frac{2E}{\hbar\omega} = 2n+1$$

i) the normalisation constant N_n

$$N_n = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} (2^n n!)^{-1/2} \rightarrow (16)$$

ii) exponential factor $e^{-y^2/2}$

iii) a polynomial $H_n(y)$ called Hermite polynomial in either odd or even powers of y

The general formula for the n^{th} wave function

$$\psi_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} (2^n n!)^{-1/2} e^{-y^2/2} H_n(y) \rightarrow (17)$$

Significance of zero point energy

For lowest state, $n=0$

$$E_0 = \frac{1}{2} h\nu$$

In old quantum mechanics, the energy n^{th} level

$$E_n = nh\nu$$

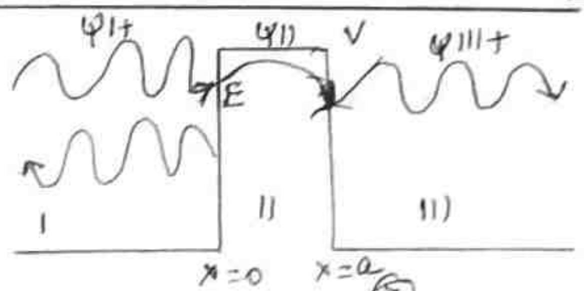
wave mechanics

$$E_n = \left(n + \frac{1}{2}\right) h\nu$$

② Discuss barrier penetration and quantum Tunneling (Qualitative)

- According to classical ideas, a particle striking a hard wall has no chance of leaking through it.
- The behaviour of a quantum particle is different due to the wave nature associated with it.
- Electromagnetic wave strikes at the interface of two media, it is partly reflected and partly transmitted.
- De-Broglie wave also partly reflected from the boundary of the potential well and partly penetrating through the barrier
- quantum mechanics leads to an entirely new result.
- It shows that there is a finite chance for the electron to leak to the other side of the barrier.
- The electron tunneled through the potential barrier and hence in quantum mechanics, this phenomenon is called tunneling.
- The transmission of electrons through the barrier is known as barrier penetration.

Expression for Transmission Probability



③

- The particle in region I has certain Probability of passing through the barrier to reach region II and then emerge out on the other side in region III
- The particle lacks the energy to go over the top of the barrier, but tunnels through it.
- Consider a beam of identical particles, all having Kinetic energy E.
- The beam is incident on the potential barrier of height V and width a from region I.
- On both sides of the barrier $V=0$. This means that no forces act on particles in regions I and III
- ψ_1 - represents the particle moving towards the barrier from region -I, while ψ_1 - represents the particle reflected moving away from the barrier.
- wave function ψ_{II} represents the particle inside the barrier.
- Some of the particles end up in region III while the others return to region I.

$$T = \frac{\text{Number of Particles transmitted}}{\text{Number of Particles incident}}$$

This probability is approximately given by

$$T = T_0 e^{-2ka}$$

where

$$k = \frac{\sqrt{2m(V-E)}}{h}$$ and a is the width of the barrier.

T_0 - Constant close to unity.

- Probability of particle penetration through a potential barrier depends on the height and width of the barrier.

Significance of the study of barrier Penetration Problems

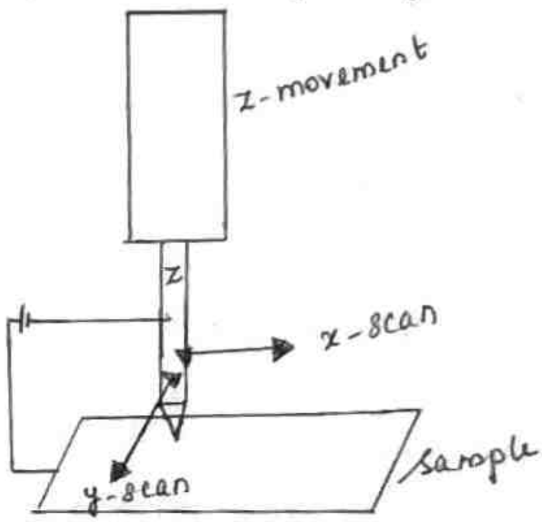
- Tunnelling is a very important physical phenomena which occurs in certain semiconductor diodes.
- The tunneling effect also occurs in the case of the alpha Particles.
- The ability of electrons to tunnel through a potential barrier is used in the Scanning Tunneling Microscope to study surfaces on an atomic scale of size.

③ What is the principle of scanning tunneling microscope. Explain the construction and working scanning tunneling microscope with a suitable diagram.

A scanning tunneling microscope or STM is a type of electron microscope.
- Commonly used in fundamental and industrial research.

Principle

It is based on the concept of quantum mechanical tunneling of electrons.
- A sharp narrow conducting needle or tip is brought very near to the surface to be examined.
- A small voltage difference about 1V is applied between the tip and the surface of the material.
- This allows electrons to tunnel through the vacuum between them and results in tunneling current.
- Information about surface morphology is obtained by monitoring the tunneling current.
- The tip's position scans across the surface and it is usually displayed in image form.



Construction

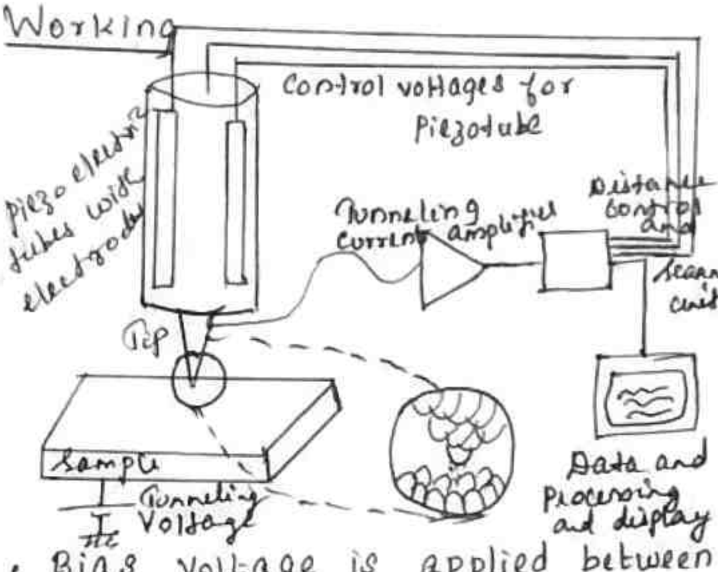
Components

i) Scanning needle tip
ii) Piezoelectric controlled height and surface (x, y) scanner
iii) Coarse sample to tip control

4. Vibration isolation system
5. Computer

- Needle tip made of tungsten
- Piezoelectric tube is provided
- Moving x, y, z directions.
- Coarse sample to tip control is used to bring the tip close to the sample.
- Any vibration
- Acquire data
- quantitative measurement.

Working



- Bias voltage is applied between the sample and the tip.
- when the needle is in positive potential, electrons can tunnel through the gap and set up a small tunneling current.
- tunneling current is amplified and measured.
- with the help of tunneling current, the feedback electronics keeps the distance between tip and sample constant.
- Once tunneling is established, sample can be verified and data are obtained.

Scanning

- If the tip is moved across the sample in the x-y plane, changes are mapped in images to present the surface morphology.
- The height z of the tip corresponding to a constant current be measured.

Advantages of STM

- For an STM, good resolution is 0.1 nm lateral resolution and 0.01nm depth resolution.
- To examine surfaces at an atomic level.
- STMs are also versatile.
- Used in ultra high vacuum, air, water and other liquids and gasses.

Disadvantages of STM

- STMs can be difficult to use effectively.
- A small vibration even a sound, can disturb the tip and the sample together.
- Even a single dust particle can damage the needle.

Applications of STM

- It is a powerful tool used in many research fields and industries to obtain atomic sample imaging and magnification

- STM recently found manipulation of atoms.
- It is used to analyze the electronic structures of the active sites at catalyst surfaces.
- STM is used in the study of structure, growth, morphology electronic structure of surface, thin films and nano structures.

(4) Write a note on Resonant Diode

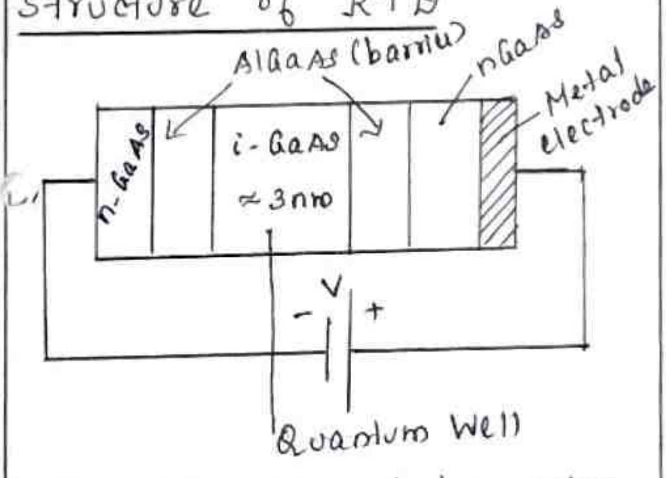
Definition

A resonant tunneling diode is a diode with resonant tunneling structure. The electrons can tunnel through some resonant states at certain energy levels.

Principle :

When electron incident with energy equal to energy level of a potential well of thin barrier, then the tunneling reaches its maximum value. This is known as resonant tunneling.

Structure of RTD



- Structure is made by using n type GaAs for the regions to the left and right of both barriers (regions 1 & 5)
- Tunneling is controlled by applying a bias voltage across the device.

Working

Tunneling Control

By applying a bias voltage across the device.

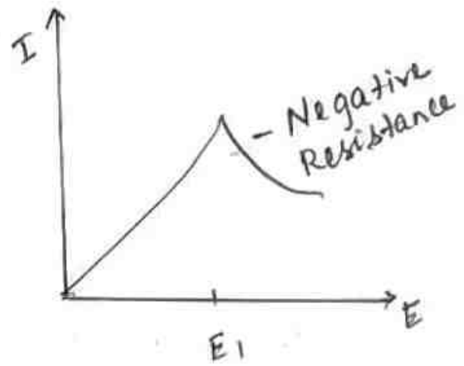
With out applied bias

- No applied bias.
- Very difficult to control the barrier height as well as the width of the potential well to match the energy of the electron.

With applied bias

- when voltage is applied, the band diagram shifts.
- Voltage is verified.
- Potential well matches with the energy of the electron wave.

Current - Energy Characteristic for a resonant tunneling diode.



- Incident electron energy E is very different stat E_n.
- transmission is low.
- E tends to E_n, transmission will increase, becoming a maximum when $E = E_n$.
- E increases tunneling will increase, reaching a peak, $E = E_1$.
- After that point E will result a decreasing current.
- Decrease of current with an increase of bias is called negative resistance

Application and uses of Resonant Tunneling Diodes

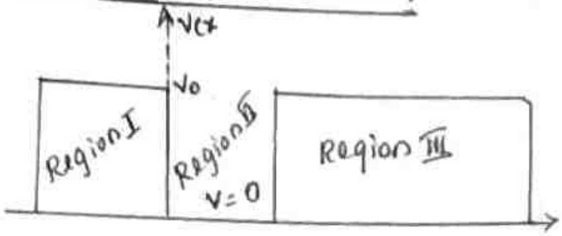
- Very good rectifiers.
- used in digital logic circuits.
- used in inverters, memory cells, and transistors.

Advantages

- Very Compact
- They are capable of ultra high speed operations because the quantum tunneling effect through the very thin layers is a very fast process.

Discuss a

(5) Particle in a finite Potential well starting from schrodinger wave eqn (Qualitative)



· Consider a particle of mass m
· x-direction between $x=0$ and $x=a$.

Step : I

E - Total energy of particle
& V - Potential Energy.
Potential energy is assumed to be zero with in the box.

$V(x) = V_0 \quad x \leq 0$ Region I

$V(x) = 0 \quad 0 < x < a$ Region II

$V(x) = V_0 \quad x \geq a$ Region III

· $E < V_0$ cannot be Present in regions I and III outside the box.

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0 \longrightarrow (1)$$

Step: II

Three regions I, II, III separately, let $\psi_I, \psi_{II}, \psi_{III}$ be the wave functions.

region I

$$\frac{d^2\psi_1}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_1 = 0 \longrightarrow (2)$$

For region II

$$\frac{d^2\psi_{II}}{dx^2} + \frac{2mE}{\hbar^2}\psi_{II} = 0 \longrightarrow (3)$$

For region III

$$\frac{d^2\psi_{III}}{dx^2} + \frac{2m}{\hbar^2}(E-V_0)\psi_{III} = 0 \longrightarrow (4)$$

$$\frac{2mE}{\hbar^2} = k^2 \text{ and } \frac{2m(E-V_0)}{\hbar^2} = -k'^2$$

$$(\text{as } E < V_0) \longrightarrow (5)$$

The eqn in the three regions written as

$$\left.\begin{aligned} &\frac{d^2\psi_1}{dx^2} - k'^2\psi_1 = 0 \\ &\frac{d^2\psi_{II}}{dx^2} + k^2\psi_{II} = 0 \\ &\frac{d^2\psi_{III}}{dx^2} - k'^2\psi_{III} = 0 \end{aligned}\right\} \longrightarrow (6)$$

Step: III

$\psi_1 = Ae^{k'x} + Be^{-k'x}$ for $x<0$

$\psi_{II} = P.e^{ikx} + Q.e^{-ikx}$ for $0<x<a$

$\psi_{III} = C.e^{k'x} + De^{-k'x}$ for $x>a$

Step : IV : As $x \to \pm\infty$, ψ should not become infinite. Hence $B=0$ and $C=0$

(9)

The wave functions in three regions

$$\psi = Ae^{k'x}$$

$$\psi_{II} = P.e^{ikx} + Q.e^{-ikx}$$

$$\psi_{III} = D.e^{-k'x}$$

Step V :

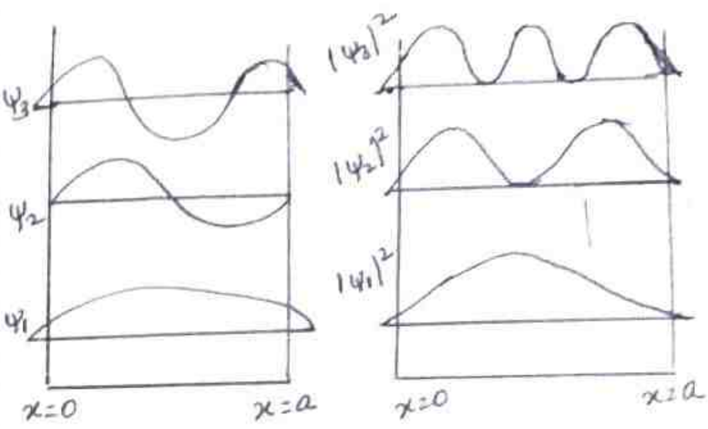
The wave function ψ and its derivative $\frac{d\psi}{dx}$ should be Continuous in the region where ψ is defined.

$$\psi_I(0) = \psi_{II}(0)$$

$$\left[\frac{d\psi_I}{dx}\right]_{x=0} = \left[\frac{d\psi_{II}}{dx}\right]_{x=0}$$

$$\psi_{II}(a) = \psi_{III}(a)$$

$$\left[\frac{d\psi_{II}}{dx}\right]_{x=a} = \left[\frac{d\psi_{III}}{dx}\right]_{x=a} \rightarrow (8)$$



a) wave functions

b) Probability densities inside non-rigid box

ict

⑥ Explain Bloch's theorem for Particles in a Periodic Potential

Bloch theorem

It is a mathematical statement regarding the form of one electron wave function for a perfectly periodic potential.

Statement

If an electron in a linear lattice of lattice constant 'a' characterised by a potential function

$V(x) = V(x+a)$ satisfies the Schrodinger eqn

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}\left[E - V(x)\right]\psi(x) = 0 \rightarrow (1)$$

(10)

then the wave functions $\psi(x)$ of electron (with energy E) is obtained as a solution of Schrodinger eqn are of the form

$$\psi(x) = U_k(x)\, e^{ikx} \quad \rightarrow (2)$$

$$U_k(x) = U_k(x+a) \quad \rightarrow (3)$$

$U_k(x)$ is also periodic with lattice periodicity.

The potential $V(x)$ is periodic as $V(x) = V(x+a)$ where a is a lattice constant.

The solutions are plane waves modulated by the function $U_k(x)$ which has the same periodicity as the lattice.

This theorem is known as Bloch theorem.

Proof :

we can write the property of the Bloch functions eqn (3)

$$\psi(x+a) = e^{ik(x+a)}\, U_k(x+a)$$

$$\psi(x+a) = e^{ikx}\, e^{ika}\, U_k(x+a)$$

Since $\psi(x) = e^{ikx} U_k(x)$

$$\psi(x+a) = e^{ika} \cdot \psi(x) \quad \rightarrow (5)$$

(or)

$$\psi(x+a) = Q\,\psi(x) \quad \rightarrow (6)$$

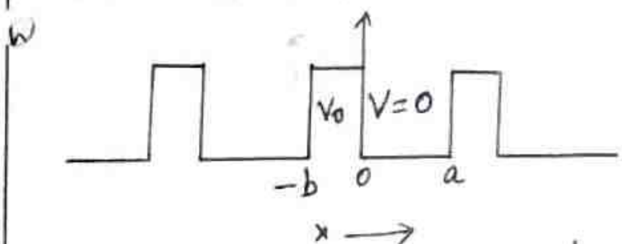
where

$$Q = e^{ika}$$

$\psi(x)$ is a single valued function

$\psi(x) = \psi(x+a)$. Thus Bloch theorem is proved.

7) Discuss of Kraning Penney model



. It was first discussed by kronig and Penny in the year 1931.

. Behaviour of electronic potential is studied by considering a periodic rectangular well structure in one dimension.

. $0 < x < a$, the potential energy is zero

. $-b < x < 0$, the potential energy is V_0.

The one dimensional Schrodinger wave eqn for two regions are written as

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E-0]\psi = 0 \quad \text{for } 0<x<a \quad \rightarrow (1)$$

$$\frac{d^2\psi}{dx^2} + \alpha^2\psi = 0 \longrightarrow (2)$$

$$\alpha^2 = \frac{2mE}{\hbar^2}$$

and

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi = 0$$

for $-b < x < 0$ $\rightarrow (3)$

$$\frac{d^2\psi}{dx^2} - \beta^2\psi = 0 \longrightarrow (4)$$

$$\beta^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

For both the regions the appropriate solution suggested by Bloch is of the from

$$\psi = e^{ikx} U_k(x) \longrightarrow (5)$$

Differentiating eqn (5) and Substituting in eqn (2) and (4) and further solving it under boundary Conditions

$$P\frac{\sin\alpha a}{\alpha a} + \cos\alpha a = \cos ka \longrightarrow (6)$$

where

$$\alpha = \frac{\sqrt{2mE}}{\hbar}, \quad P = \frac{mV_0ba}{\hbar^2}$$

The term P is called as scattering Power of the Potential barrier.

. It is a measure of strength with which the electrons are attracted by the positive ions

From the graph, $P \to 0$

$$\cos\alpha a = \cos ka$$

$$\alpha = k, \quad \alpha^2 = k^2$$

$$\frac{2mE}{\hbar^2} = k^2 \qquad \left[\because \hbar = h/2\pi\right]$$

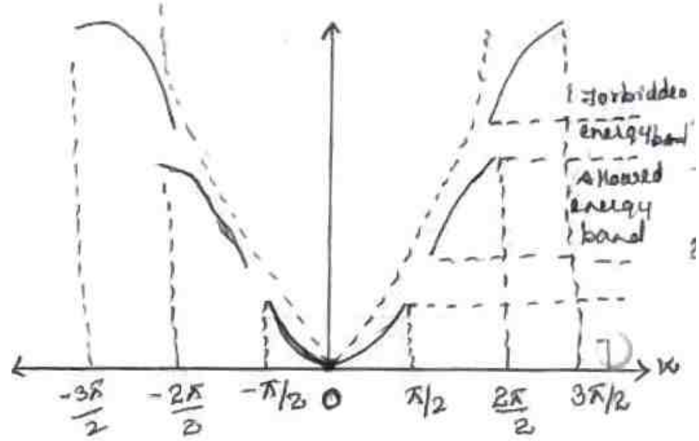
$$E = \frac{\hbar k^2}{2mE}$$

$$\boxed{E = \frac{h^2k^2}{8\pi^2 m}}$$

E-k curve

The energy of the electron in the periodic lattice

$$\boxed{E = \frac{h^2k^2}{8\pi m} \cdot k^2}$$



(8) Describe origin of energy bands in solid

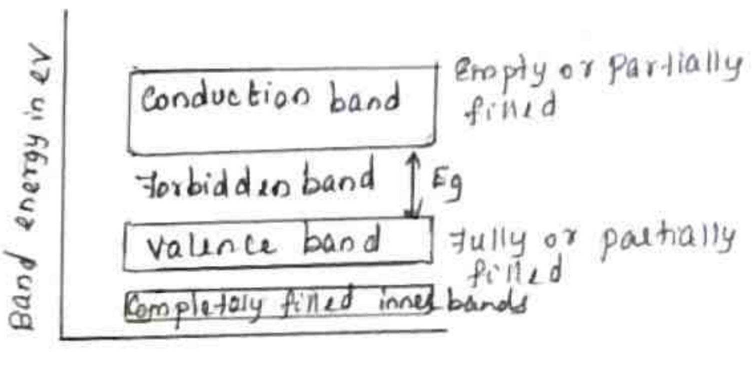
- All the atoms of a solid, isolated from one another, can have completely identical electronic schemes of their energy levels.
- Electrons fill the levels in each atom independently.
- Closely spaced energy levels known as permitted energy band.
- Lower completely filled band is valence band.
- Upper unfilled band is called conduction band.

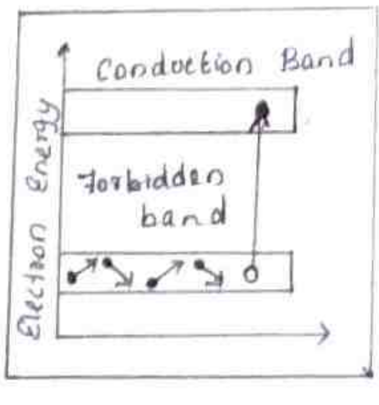
Definition

A set of such closely spaced energy levels is called an energy band.

Concept of valence band, conduction band and forbidden band

- An electron in a solid can have only discrete energies that lie within these energy bands. These bands are called allowed energy bands.
- Band corresponding to valence electron is called valence band.
- Band beyond forbidden band is called conduction band.
- Electrons in the outermost shell are called valence electrons.
- No allowed energy levels in some gaps called forbidden energy bands.



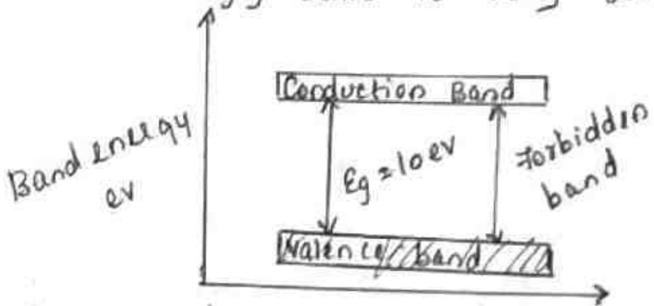


Classification of Metals, Semiconductors and insulators

Solids are classified into insulators, semiconductors and conductors.

Insulators

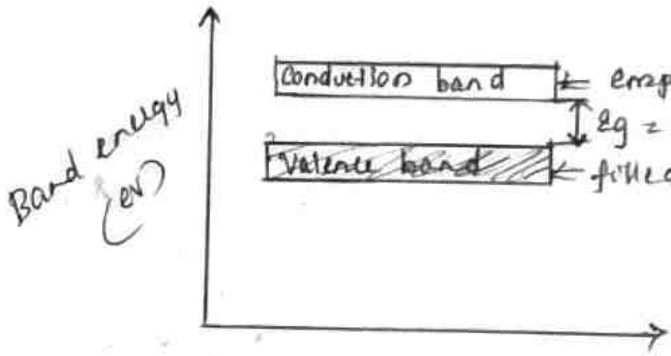
- Energy gap between conduction band and valence band is very high about 10 ev
- Forbidden energy band is very wide.



- Conduction band is completely vacant and valence band is completely filled.

Semiconductors

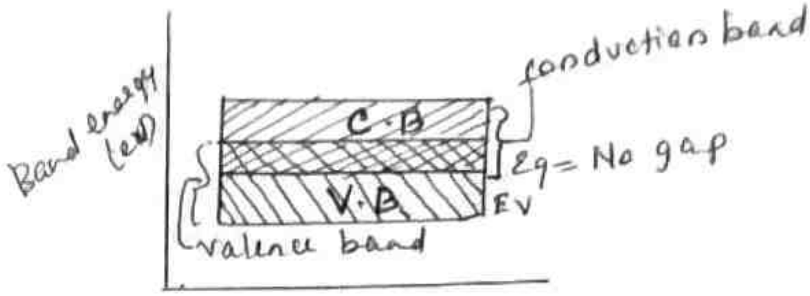
- Forbidden gap is very small
- Examples : Germanium and silicon
- Energy gap between conduction band and valence band is very small
- 0.5 ev to 1ev



- Conduction band is partially filled and valence band is partially vacant.

Conductor :

- No forbidden gap
- Both valence and conduction bands overlap each other



- As the temperature increases, the electrical conduction decreases
- Mobility decreases due to large number of collisions with ions.